A (q,t)-OVERVIEW OF q-ANALOGS

SEE 260B.30979
FOR MANY REFERENCES

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q-ANALOGS

$$P(q) = a_0 + a_1 q + \ldots + a_d q^d \qquad a_i \in \mathbb{N}$$

POSSIBLE PROPERTIES

- PALINDROMIC $\quad P(q) = q^d P(1/q) \qquad a_i = a_{d-i}$
- PALINDROMIC+UNIMODAL $\quad a_0 \le a_1 \le \cdots \le a_{\lfloor d/2 \rfloor}$
- γ-POSITIVE $\quad P(q) = \sum_{0 \le k \le d/2} \gamma_k \cdot q^k (1+q)^{d-2k} \qquad \gamma_k \ge 0$

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(q,t)-ANALOGS

$$P(q,t) = a_0 t^d + a_1 q t^{d-1} + \dots + a_d q^d \qquad a_i \in \mathbb{N}$$

POSSIBLE PROPERTIES

- PALINDROMIC $\Rightarrow P(q,t) = P(t,q)$

- γ-POSITIVE $P(q,t) = \sum_{0 \le 2k \le d} \gamma_k \cdot (qt)^k (q+t)^{d-2k}$

$$\gamma_k \ge 0$$

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$$[n]_{q,t} := \frac{q^n - t^n}{q - t}$$

$$= q^{n-1} + \cdots + t^{n-1}$$

$$[n]!_{q,t} := [n]_{q,t} \cdot [n-1]_{q,t} \cdots [1]_{q,t}$$

$$[0]!_{q,t} := [1]_{q,t} = 1$$

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$$[0]!_{q,t} = 1$$

$$[1]!_{q,t} = 1$$

$$[2]!_{q,t} = q + t$$

$$[3]!_{q,t} = (q+t)(q^2+qt+t^2)$$

$$= q^3+2q^2t+2qt^2+t^3$$

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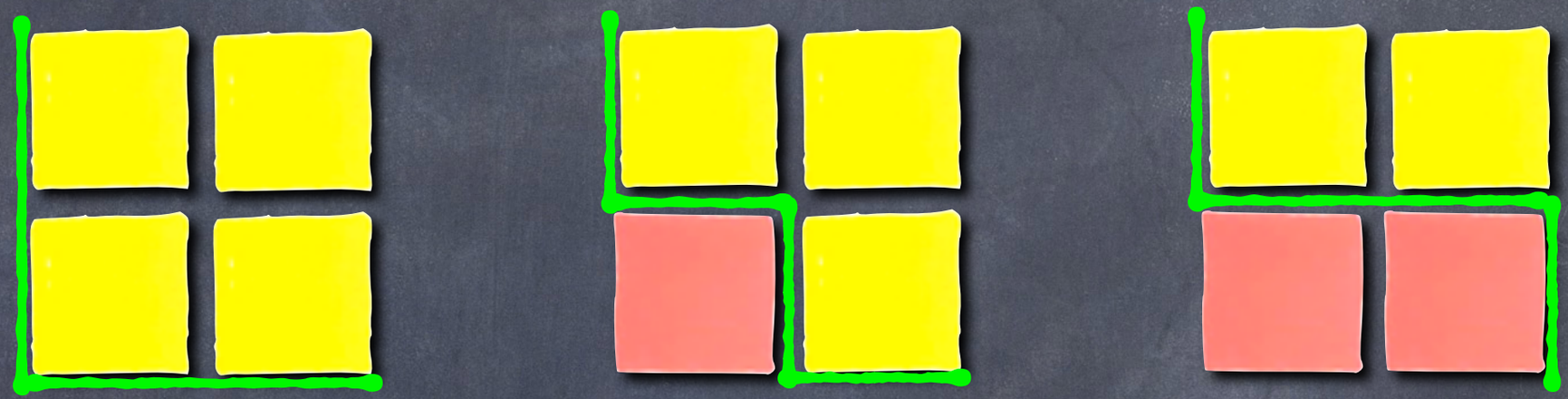
$$\begin{bmatrix} n \\ k \end{bmatrix}_{q,t} := \frac{[n]!_{q,t}}{[k]!_{q,t}[n-k]!_{q,t}}$$

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_{q,t} = q^4 + q^3 t + 2q^2 t^2 + q t^3 + t^4$$

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$$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_{qt} = q^4 + q^3t + 2q^2t^2 + qt^3 + t^4 \quad \in \mathbb{N}[q,t]$$



q^4 q^3t q^2t^2



q^2t^2 qt^3 t^4

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$$\begin{bmatrix} n+k \\ k \end{bmatrix}_{q,t} = \sum_{\substack{|w|_0 = n \\ |w|_1 = k}} q^{\mathrm{inv}(w)} t^{n \cdot k - \mathrm{inv}(w)}$$

$$[n]!_{q,t} = \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} t^{\binom{n}{2} - \mathrm{inv}(\sigma)}$$

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(q,t)- VANDERMONDE

$$\begin{bmatrix} a+b \\ r \end{bmatrix}_{qt} = \sum_{i+j=r} q^{j(a-i)} t^{i(b-j)} \begin{bmatrix} a \\ i \end{bmatrix}_{qt} \begin{bmatrix} b \\ j \end{bmatrix}_{qt}$$

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(q,t)-Eulerian Polynomials

$$A_m(q,t) = \sum_{\sigma \in \mathfrak{S}_m} q^{\mathrm{DES}(\sigma)} t^{m-1-\mathrm{DES}(\sigma)}$$

$$A_1(q,t) = 1$$

$$A_2(q,t) = q + t$$

$$A_3(q,t) = q^2 + 4qt + t^2$$

$$A_4(q,t) = q^3 + 11q^2t + 11qt^2 + t^3$$

$$A_5(q,t) = q^4 + 26q^3t + 66q^2t^2 + 26qt^3 + t^3$$

$$\vdots$$

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(q,t)-CATALAN

$$C_n(q,t) := \frac{1}{[n+1]_{qt}} \begin{bmatrix} 2n \\ n \end{bmatrix}_{qt}$$

(q,t)-NARAYANA

$$N_{nk}(q,t) := \frac{1}{[k]_{qt}} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{qt} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{qt}$$

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Rational (q,t)-Catalan

$$\frac{1}{[a+b]_{qt}} \begin{bmatrix} a+b \\ a \end{bmatrix}_{qt}$$

$$(a,b)=1$$

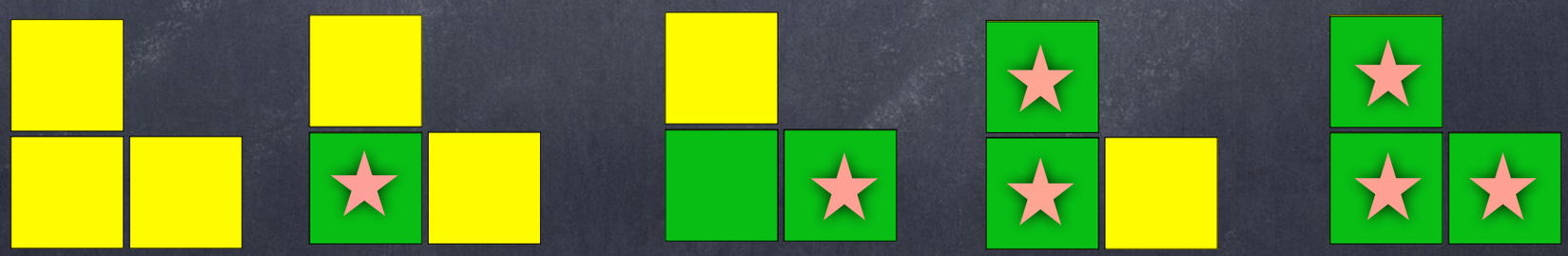
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ANOTHER (q,t)-CATALAN

$$A_\tau(q,t) := \sum_{\alpha \subseteq \tau} q^{|\tau| - |\alpha|}\, t^{\mathrm{dinv}_\tau(\alpha)}$$

$$A_{\square\!\square\!\square}(q,t) = q^3 + q^2 t + q t + q t^2 + t^3$$



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GRADED RING OF SYMMETRIC FUNCTIONS

$$\Lambda = \bigoplus_d \Lambda_d$$

DEGREE d HOMOGENEOUS COMPONENT

$$f(x_1, x_2, x_3, \dots) \in \Lambda_d$$

$$f(q, t, 0, \dots) \in \Lambda_d^{(2)} \quad [q, t]$$

(PALINDROMIC)

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SYMMETRIC FUNCTIONS

$$\Lambda_{ab}(q,t) := q^a t^b + \cdots + q^b t^a \qquad a \geq b \geq 0$$

$$= (qt)^b \, [a-b+1]_{qt}$$

$$h_n(q,t) = [n+1]_{qt}$$

$$e_1(q,t) = q + t \qquad e_2(q,t) = qt$$

$$e_a(q,t) = 0 \qquad a > 2$$

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Bases of $\Lambda^{(2)}_d$

$$\{\Lambda_{ab}\}_{a+b=d}$$

$$\{e_1^a e_2^b\}_{a+2b=d}$$

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$$[n] = [n]_{qt} \qquad [n]! = [n]!_{qt} \qquad p_n := q^n + t^n$$

$$[n+1] = \sum_{0 \le 2k \le n} \binom{n-k}{k} e_1^{n-2k} (-e_2)^k$$

$$[n]! \in \mathbb{N}[e_1, -e_2]$$

$$p_n \in \mathbb{N}[e_1, -e_2]$$

$$p_n = \sum_{0 \le 2k \le n} \frac{k}{n-k} \binom{n-k}{k} e_1^{n-2k} (-e_2)^k$$

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Plethysm in $\Lambda^{(2)}_d$

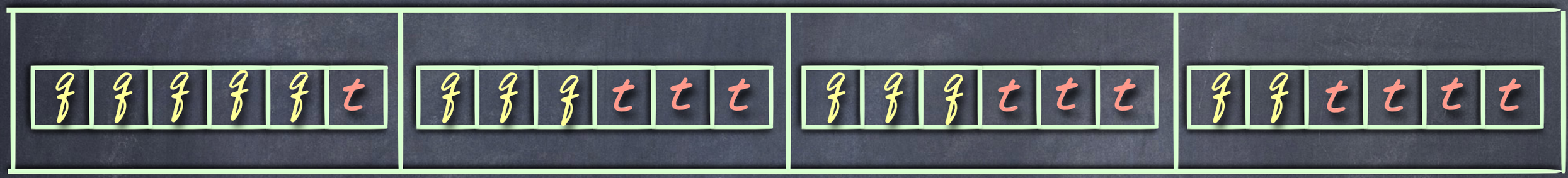
$$g^2 t^4$$

$$h_k(q,t) = \cdots + \underbrace{\boxed{q}\boxed{q}\boxed{t}\boxed{t}\boxed{t}\boxed{t}}_{k} + \cdots$$

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$$h_M \circ h_K$$



M

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$$h_m \circ h_k$$

f	f	t	t	t	t

f	f	f	t	t	t

f	f	f	t	t	t

f	f	f	f	f	t

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$$h_m \circ h_k$$



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$$h_n \circ h_k(q,t) = \begin{bmatrix} n+k \\ k \end{bmatrix}_{qt}$$

$$\Delta_\mu \circ h_k(q,t) = \Delta_\mu(q^k, \ldots, t^k)$$

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WE DERIVE

- IDENTITIES
- FORMULAS
- PROPERTIES

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Hook Content Formula

$$s_\lambda \circ h_{k-1} = (qt)^{n(\lambda)} \prod_{(i,j)\in\lambda} \frac{[k+i-j]_{qt}}{[\epsilon(i,j)]_{qt}}$$

$(i,j)\in\lambda$ (i,j) cell of λ

$\epsilon(i,j)$ hook length

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SYMMETRIC (q,t)-ANALOGS

POSSIBLE PROPERTIES

- UNIMODAL $\quad P(q,t) \in \mathbb{N}\{\Delta_{ab}\}_{a+b=d}$

$\Leftrightarrow$ SCHUR-POSITIVITY

- γ-POSITIVE $\quad P(q,t) \in \mathbb{N}\{e_1^a e_2^b\}_{a+2b=d}$

$\Leftrightarrow$ e-POSITIVITY

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SCHUR - POSITIVITY

FROBENIUS OF GL_2-ACTION

$$\sum_{ab} c_{ab}\, s_{ab}(f,t)$$

PROPOSITION

THE PROBABILITY OF SCHUR-POSITIVITY OF A RANDOM POSITIVE MONOMIAL SYMMETRIC FUNCTION OF DEGREE d IS

$$\frac{1}{\lfloor d/2 \rfloor !}$$

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$$A_{\square\!\square}(q,t) = \underbrace{q^3 + q^2 t + q t^2 + t^3} + q t$$

$$= s_3(q,t) + s_{111}(q,t)$$

CONJECTURE

FOR ALL TRIANGULAR PARTITIONS τ,

$A_\tau(q,t)$ IS SCHUR POSITIVE.

(SOME CASES SHOWN)

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e-Positivity

(Permutation Representation)

Proposition

The probability of e-positivity of a random Schur-positive symmetric function of degree d is

$$\prod_{a+2b=d} \binom{b}{\lfloor b/2 \rfloor}^{-1}$$

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(q,t)-Eulerian Polynomials e-Positivity

$$A_2(q,t) = e_1^2 + 2e_2$$

$$A_3(q,t) = e_1^3 + 8e_2e_1$$

$$A_4(q,t) = e_1^4 + 22e_2e_1^2 + 16e_2^2$$

$$A_5(q,t) = e_1^5 + 52e_2e_1^3 + 136e_2^2e_1$$

$$A_6(q,t) = e_1^6 + 114e_2e_1^4 + 720e_2^2e_1^2 + 272e_2^3$$

$$\vdots$$

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(q,t)-Cyclotomic Polynomials

$$\frac{q^n - t^n}{q - t} = \prod_{\substack{(d|n) \\ d>1}} \Psi_d(q,t)$$

$\Psi_d(q,t)$ irreducible in $\mathbb{Q}[q,t]$

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$$\varphi_2 = q + t$$

$$\varphi_3 = q^2 + qt + t^2$$

$$\varphi_4 = q^2 + t^2$$

$$\varphi_5 = q^4 + q^3t + q^2t^2 + qt^3 + t^4$$

$$\varphi_6 = q^2 - qt + t^2$$

$$\varphi_d = \varphi_d(q,t)$$

$$\cdots$$

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$$\varphi_2 = e_1$$

$$\varphi_3 = e_1^2 - e_2$$

$$\varphi_4 = e_1^2 - 2e_2$$

$$\varphi_5 = e_1^4 - 3e_2e_1^2 + e_2^2$$

$$\varphi_6 = e_1^2 - 3e_2$$

$$\cdots$$

LEMMA For all d

$$\varphi_d \in \mathbb{N}[e_1, -e_2]$$

PROOF irreducible factors over $\mathbb{R}[f,t]$ are of the form

$$f^2 - 2\cos(\theta) ft + t^2 = e_1^2 + (2+2\cos(\theta))(-e_2)$$

$$2+2\cos(\theta) \geq 0$$

γ ANTI POSITIVE

$$\sum_{0 \le 2k \le d} \gamma_k \cdot (-qt)^k (q+t)^{d-2k} \qquad \gamma_k \ge 0$$

$$\mathbb{Q}[e_1, -e_2]$$

$$(-e_2)^k \, \psi_{\lambda_1} \psi_{\lambda_2} \cdots \psi_{\lambda_\ell}$$

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$$(-e_2)^k \varphi_{\lambda_1} \varphi_{\lambda_2} \cdots \varphi_{\lambda_\ell}$$

- PARTITIONS WITH DISTINCT PARTS
- ALTERNATING SIGN MATRICES
- PLANE PARTITIONS IN A CUBE
- STANDARD YOUNG TABLEAUX OF SHAPE μ



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CYCLOTOMIC GENERATING FUNCTION MONOÏD

$$\Phi_d := \left\{ (-e_2)^k \varphi_{\lambda_1} \varphi_{\lambda_2} \cdots \varphi_{\lambda_\ell} \;\middle|\; k + \sum_{i=1}^{\ell} \varphi(\lambda_i) = d \right\}$$

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$$\widetilde{(-)}: \mathbb{Q}[e_1, e_2] \longleftrightarrow \mathbb{Q}[e_1, e_2]$$

$$\widetilde{f}(e_1, e_2) = f(e_1, -e_2)$$

MULTIPLICATIVE

LINEAR

INVOLUTION

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$$\{n\} := \widetilde{[n]} \qquad \{n+1\} = e_1\{n\} + e_2\{n-1\}$$

$$\left\{ {n \atop k} \right\} := \widetilde{\left[{n \atop k} \right]}$$

$$\left\{ {n \atop k} \right\} = \{n+1\} \left\{ {n-1 \atop k} \right\} + e_2\{n-k-1\} \left\{ {n-1 \atop k-1} \right\}$$

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$$\varphi_6 = q^2 - qt + t^2$$

$$\cdots$$

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$$\widetilde{\varphi}_2 = q + t$$

$$\widetilde{\varphi}_3 = q^2 + 3qt + t^2$$

$$\widetilde{\varphi}_4 = q^2 + 4qt + t^2$$

LUCAS ATOMS

$$\widetilde{\varphi}_5 = q^4 + 7q^3t + 13q^2t^2 + 7qt^3 + t^4$$

$$\widetilde{\varphi}_6 = q^2 + 5qt + t^2$$

...

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$$\tilde{\varphi}_2 = e_1$$

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...

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$\widetilde{\varphi}_d \in \mathbb{Q}[e_1, e_2]$ IRREDUCIBLE*

* NOT SO FOR $\widetilde{\varphi}_7$, $\widetilde{\varphi}_{30}$ IN $\mathbb{Q}[q,t]$

$$\widetilde{\varphi}_7 = (q^3 + 5q^2t + 6qt^2 + t^3)(t^3 + 6t^2q + 5tq^2 + q^3)$$

NO OTHER FOR $n \leq 1000$

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$$\{n\} = \prod_{(d \mid n),\, d>1} \widetilde{\varphi}_d$$

$$\{n\}! = \prod_{d=2}^{n} \widetilde{\varphi}_d^{\,\lfloor n/d \rfloor}$$

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = \prod_{d=2}^{n} \widetilde{\varphi}_d^{\,\lfloor n/d \rfloor - \lfloor k/d \rfloor - \lfloor (n-k)/d \rfloor}$$

$$\frac{1}{\{a+b\}} \begin{Bmatrix} a+b \\ a \end{Bmatrix} = \prod_{d=2}^{n} \widetilde{\varphi}_d^{\,m(a,b;d)}$$

$$(a,b) = 1$$

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ATHANASIADIS

γ-POSITIVITY IN COMBINATORICS AND GEOMETRY

SAGAN & TIRRELL

LUCAS ATOMS

BILLEY & SWANSON

CYCLOTOMIC GENERATING FUNCTIONS

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