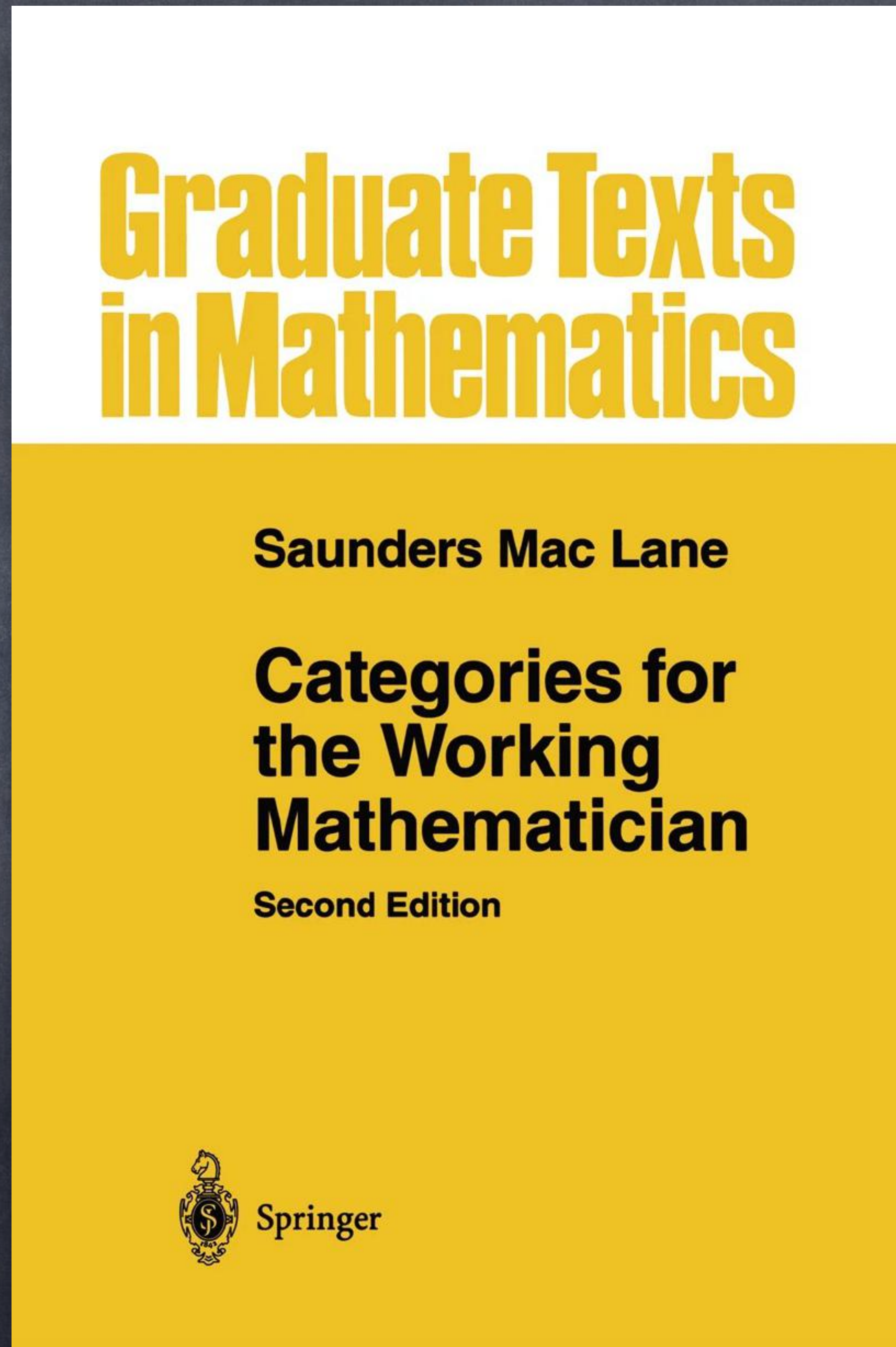
Graduate Texts in Mathematics
Saunders Mac Lane
Categories for the Working Mathematician
Second Edition
Springer

Graduate Talks in Mathematics

François Bergeron

Functors for the Working Combinatorialist

UQÀM

CATEGORIFICATION OF INTEGER SEQUENCES (GENERATING SERIES)

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Integer Sequences

$$
\begin{array}{ccc}
F : \mathbb{B} & \xrightarrow[\text{Functor}]{} & \mathbb{B} \\
\text{Card}\downarrow & & \downarrow\text{Card} \\
f : \mathbb{N} & \xrightarrow[\text{Function}]{} & \mathbb{N}
\end{array}
$$

$\mathbb{B}$: Finite sets + Bijections

$$F : \mathbb{B} \xrightarrow[\text{Functor}]{} \mathbb{B}$$

Charles Ehresmann

Species of structures

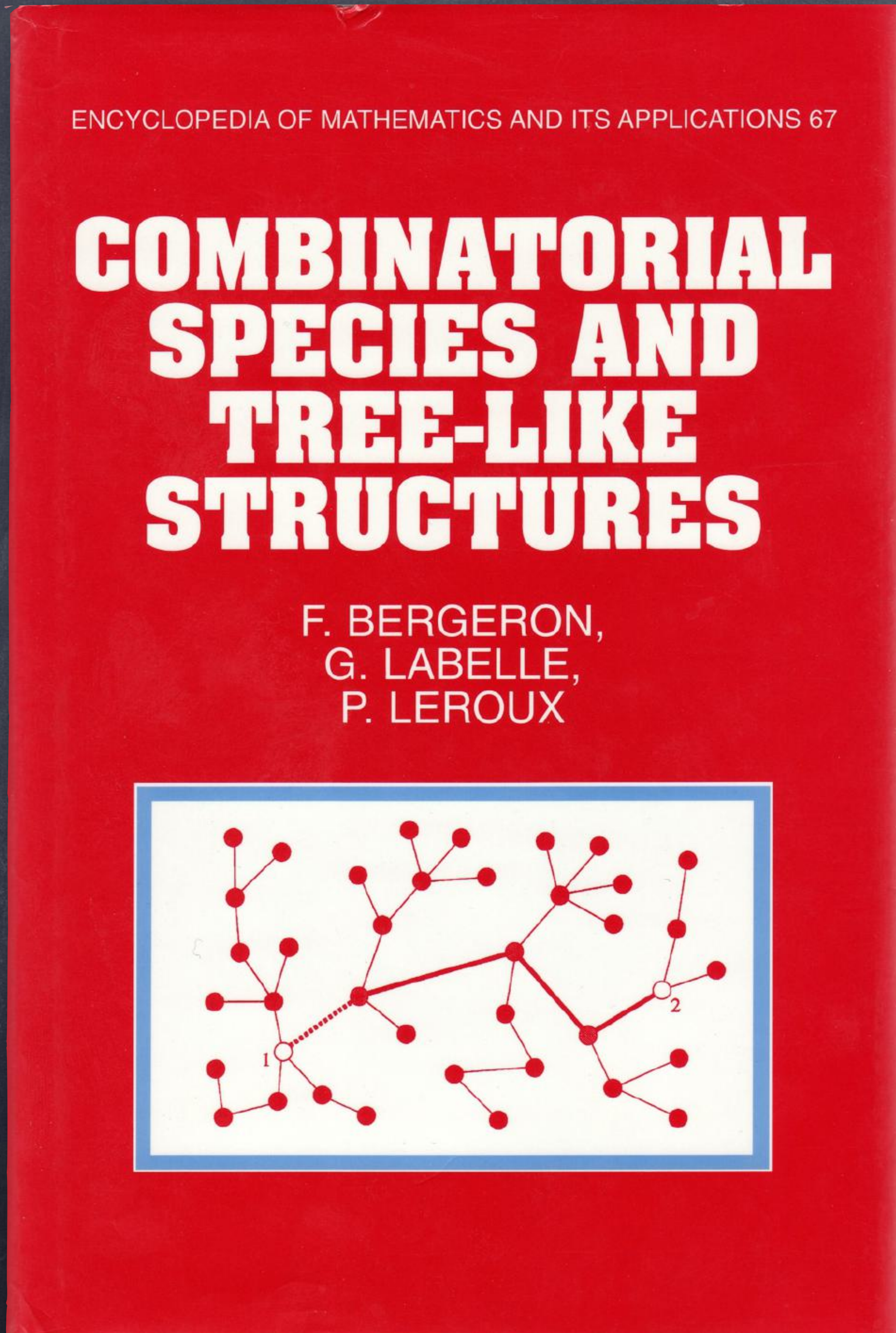
(a.k.a. : sort, kind, notion)

$F : \mathbb{B} \xrightarrow{\text{Functor}} \mathbb{B}$

André Joyal

Species of Structures

(a.k.a. : sort, kind, notion)



F.B.

GILBERT LABELLE

PIERRE LEROUX

SPECIES OF STRUCTURES

1) $A \mapsto F[A]$ STRUCTURES

2) $\varphi : A \xrightarrow{\sim} B$

$\downarrow$

$F[\varphi] : F[A] \xrightarrow{\sim} F[B]$

a) $F[Id_A] = Id_{F[A]}$

b) $F[\varphi \circ \psi] = F[\varphi] \circ F[\psi]$

TRANSPORT OF STRUCTURES

"WELL" DEFINED

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Species of structures

Examples

François Bergeron, LaCIM

Gra : The Species of Graphs

1) $\mathrm{Gra}[A]$ The set of all graphs with A as vertices

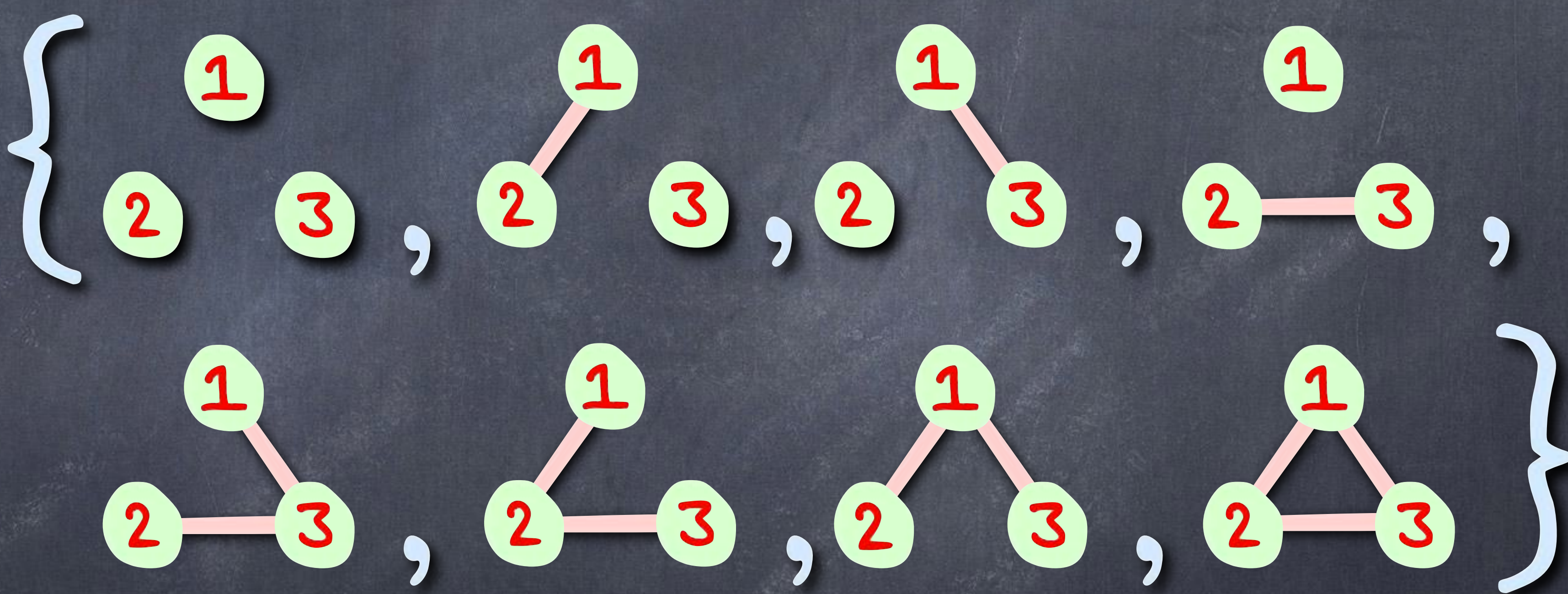
2) $\mathrm{Gra}[\varphi]:\mathrm{Gra}[A] \xrightarrow{\sim} \mathrm{Gra}[B]$

Relabel along φ

GRA : THE SPECIES OF GRAPHS

1) GRA[A] $\quad A = \{1,2,3\}$

$\mathrm{GRA}[\{1,2,3\}] =$



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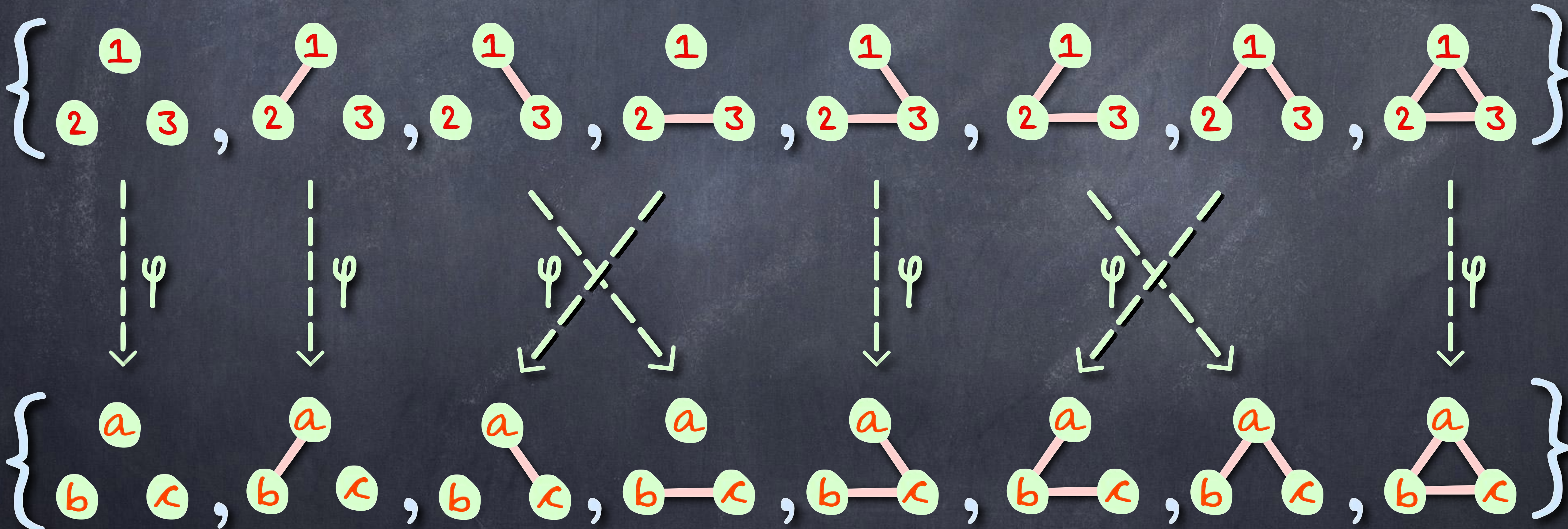
GRA : THE SPECIES OF GRAPHS

2) $\mathrm{GRA}[\varphi] : \mathrm{GRA}[A] \xrightarrow{\ \sim\ } \mathrm{GRA}[B]$

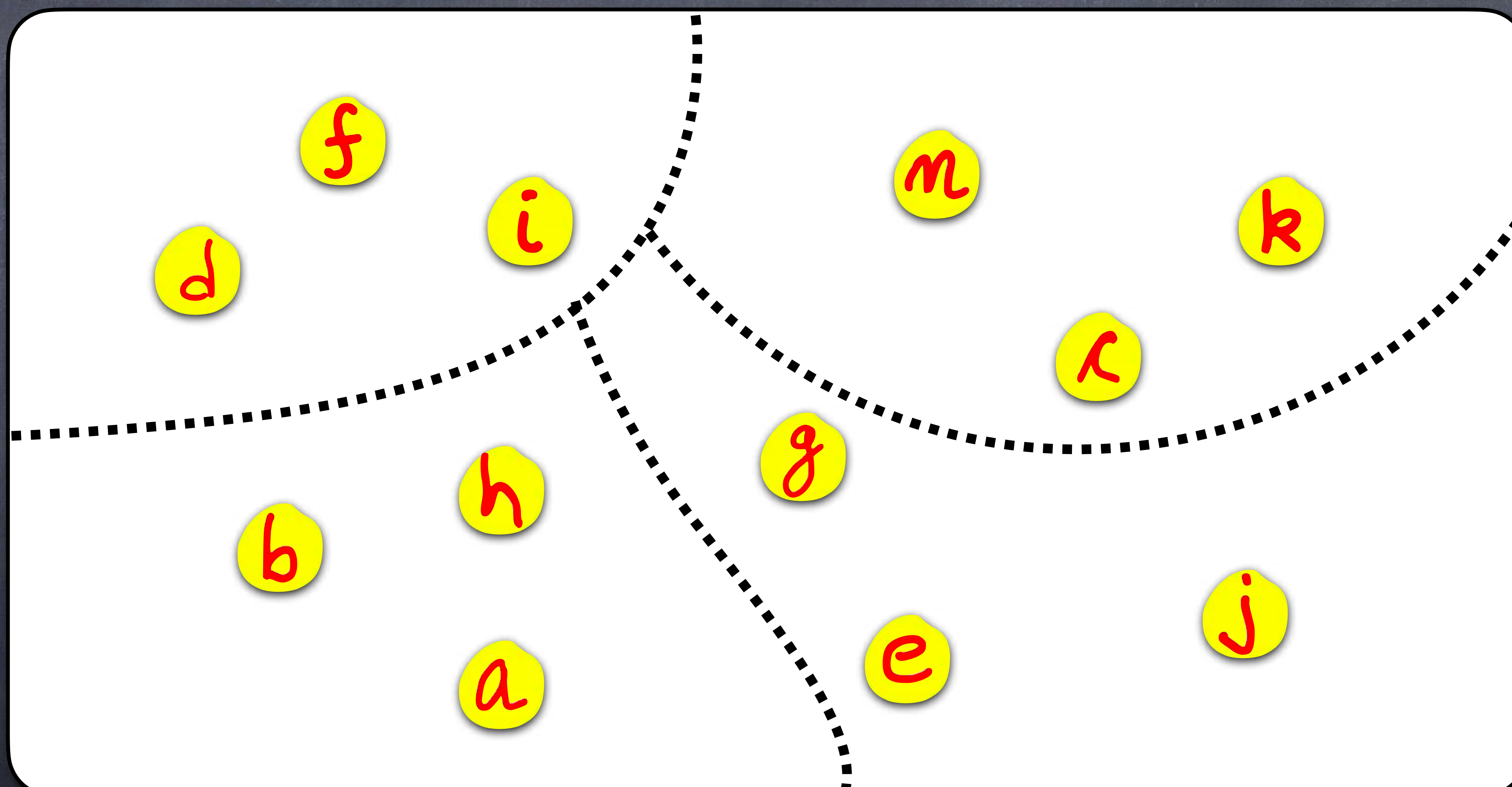
$\varphi(1) = b \qquad \varphi(2) = a \qquad \varphi(3) = c$



Part : The Species of Partitions

$\pi \in \mathrm{Part}[\{a,b,c,d,e,f,g,h,i,j,n,k\}]$

π



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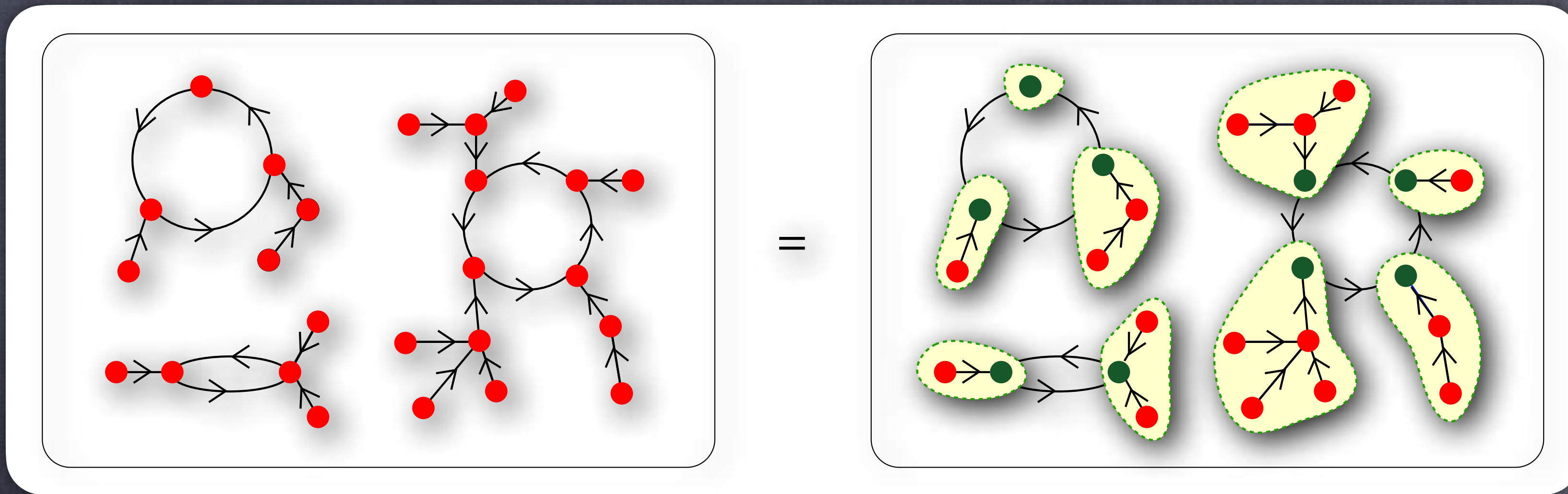
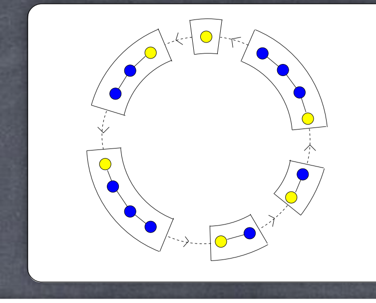
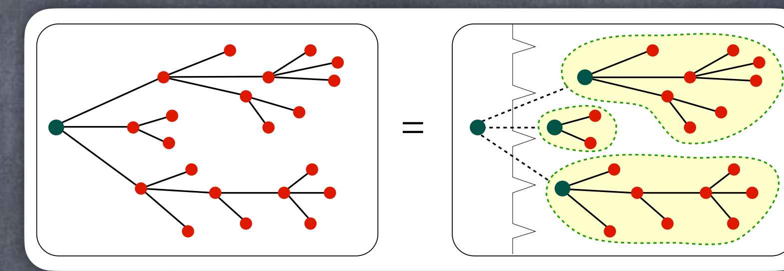
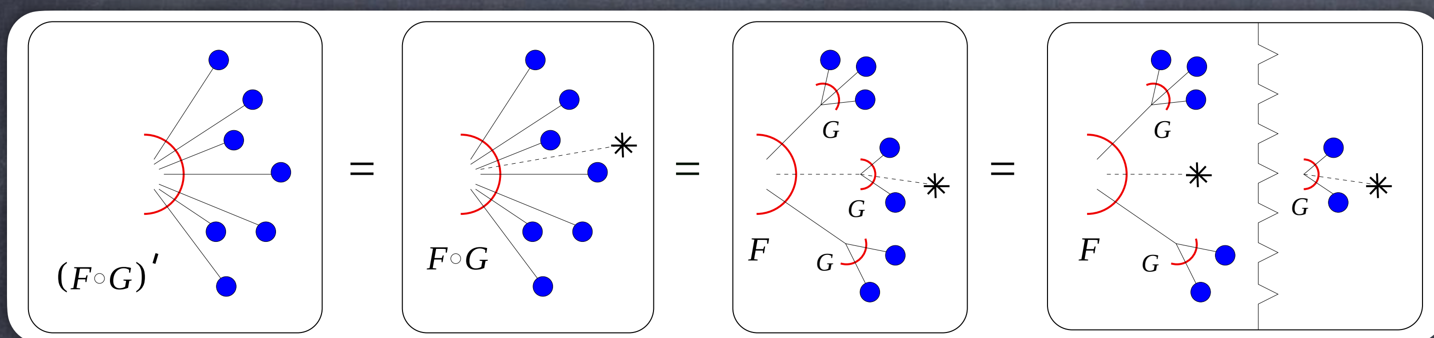
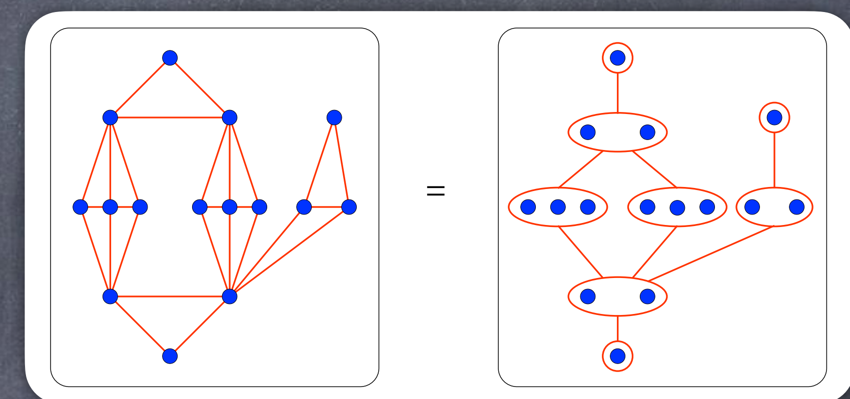
E	SETS	End	ENDOFUNCTIONS
$\mathcal{S}$	PERMUTATIONS	X	SINGLETONS
$\mathbb{L}$	LISTS	$\mathcal{T}$	ROOTED TREES
$\mathcal{C}$	CYCLES	$\mathcal{O}$	ORDERS
$\mathcal{P}$	SUBSETS	$\mathcal{D}$	DIRECTED GRAPHS
$\mathcal{T}$	TREES	$\mathcal{B}$	BINARY TREES

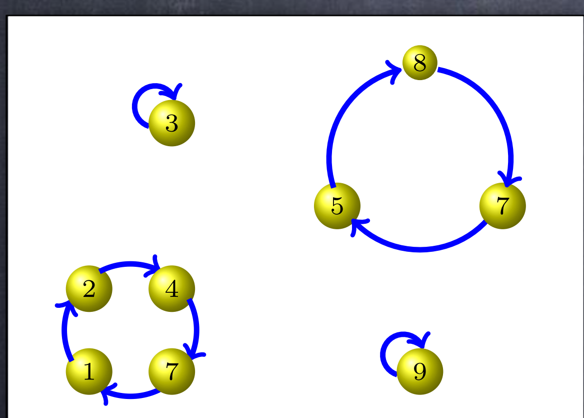


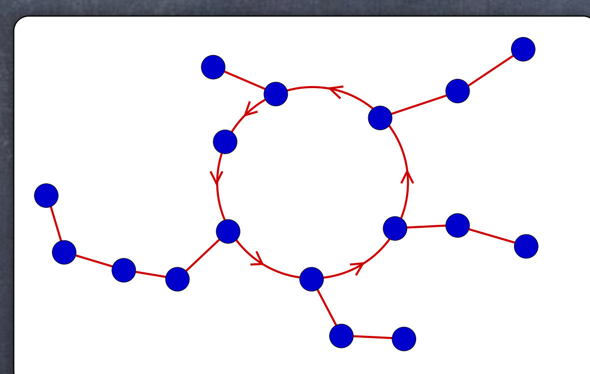


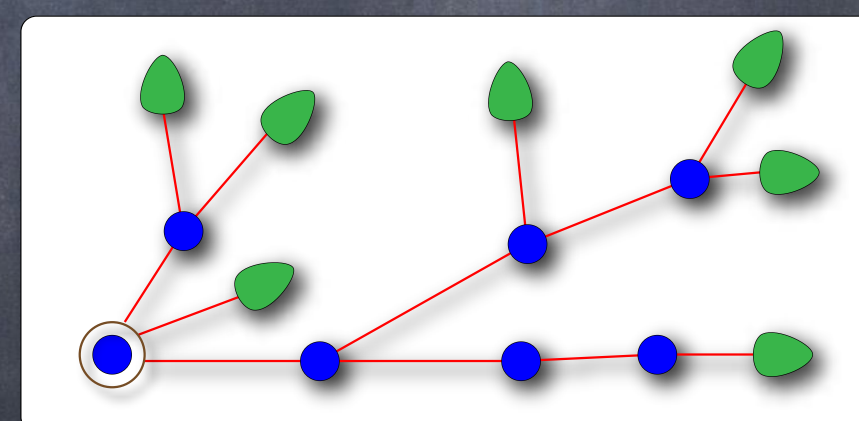


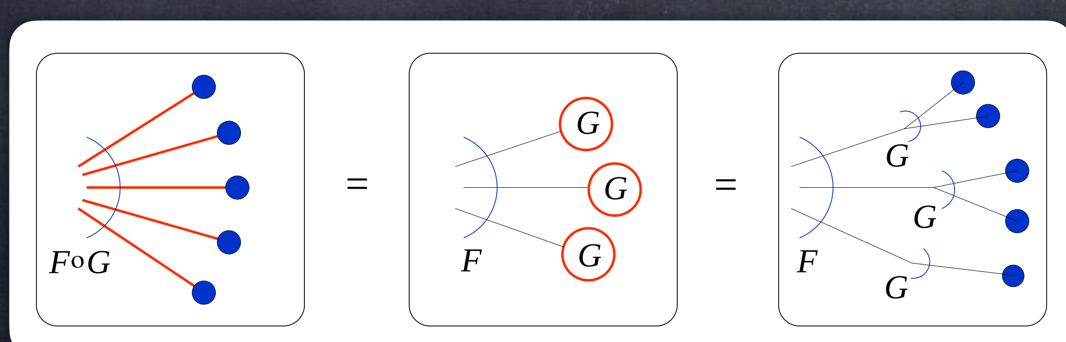


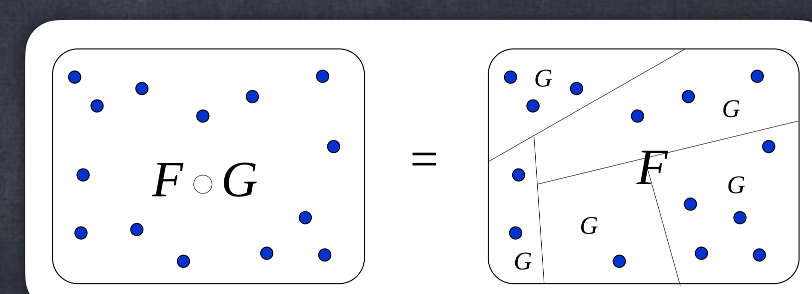












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𝔹-Spe Algebra of Species

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OPERATIONS

"+" AND "$\sum$": Disjoint union

1. $(F+G)[A] := F[A] + G[A]$

2. $(F\cdot G)[A] := \sum_{B+C=A} F[B] \times G[C]$

3. $(F\circ G)[A] := \sum_{\pi \in \mathrm{Part}[A]} F[\pi] \times \prod_{B\in\pi} G[B]$ $(G[\emptyset]=\emptyset)$

$F(G) = (F\circ G)$

4. $F'[A] := F[A+\{*\}]$

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$s \in (F+G)[A]$

s IS AN (F OR G)-STRUCTURE ON A

$s \in (F\cdot G)[A]$

s IS AN (F AND G)-STRUCTURE ON A

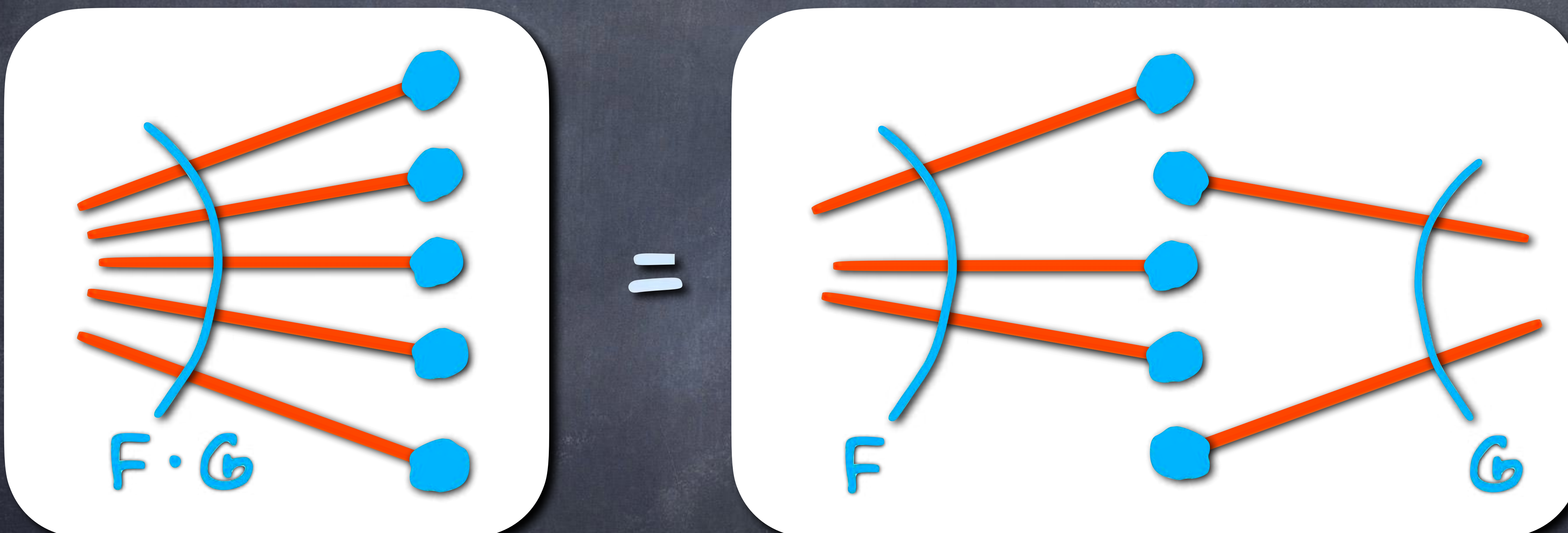
$s \in (F\circ G)[A]$

s IS AN (F OF G)-STRUCTURE ON A

Generic structure

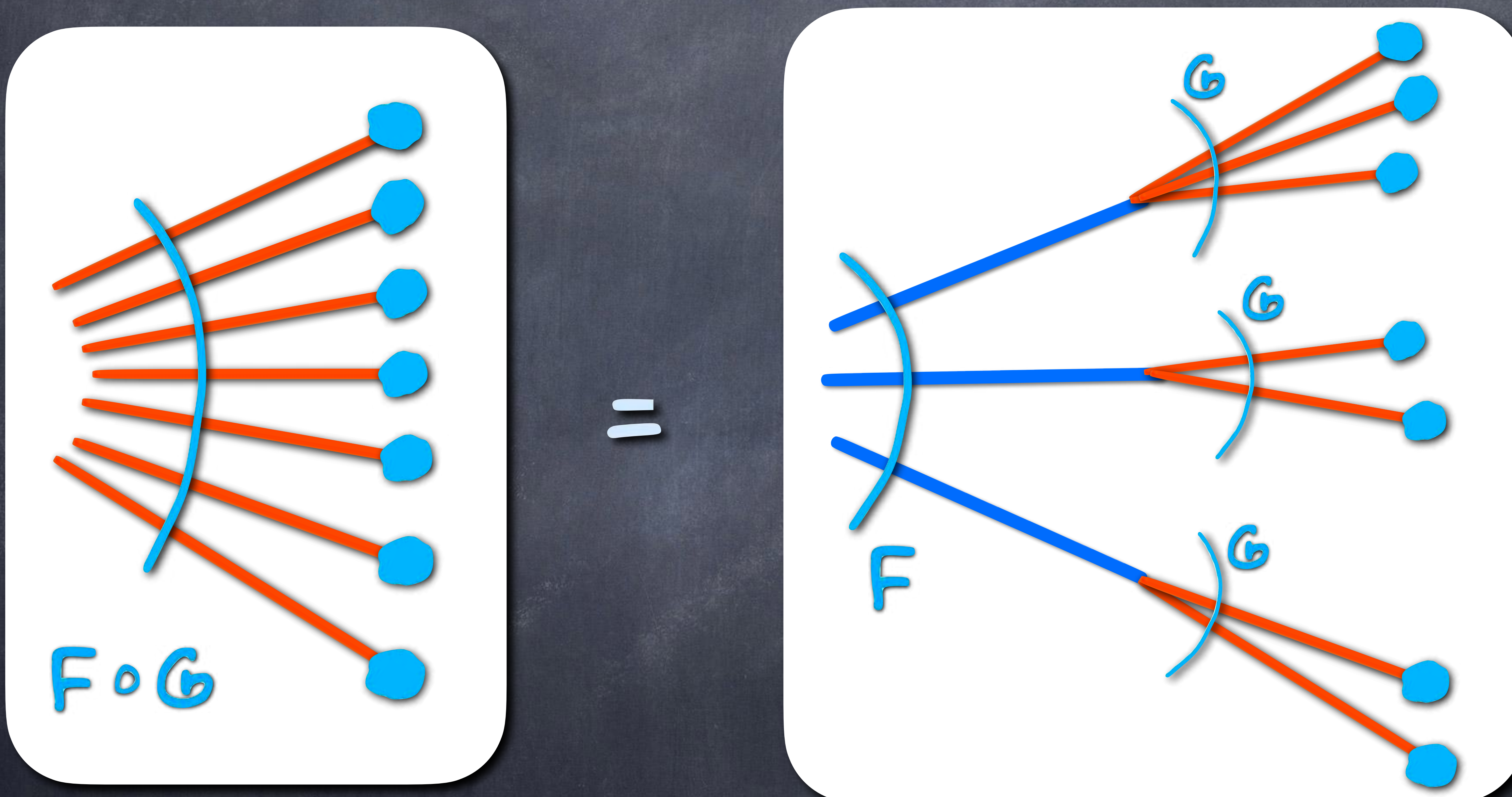
$\in F[A]$

$$(F \cdot G)[A] := \sum_{B+C=A} F[B] \times G[C]$$



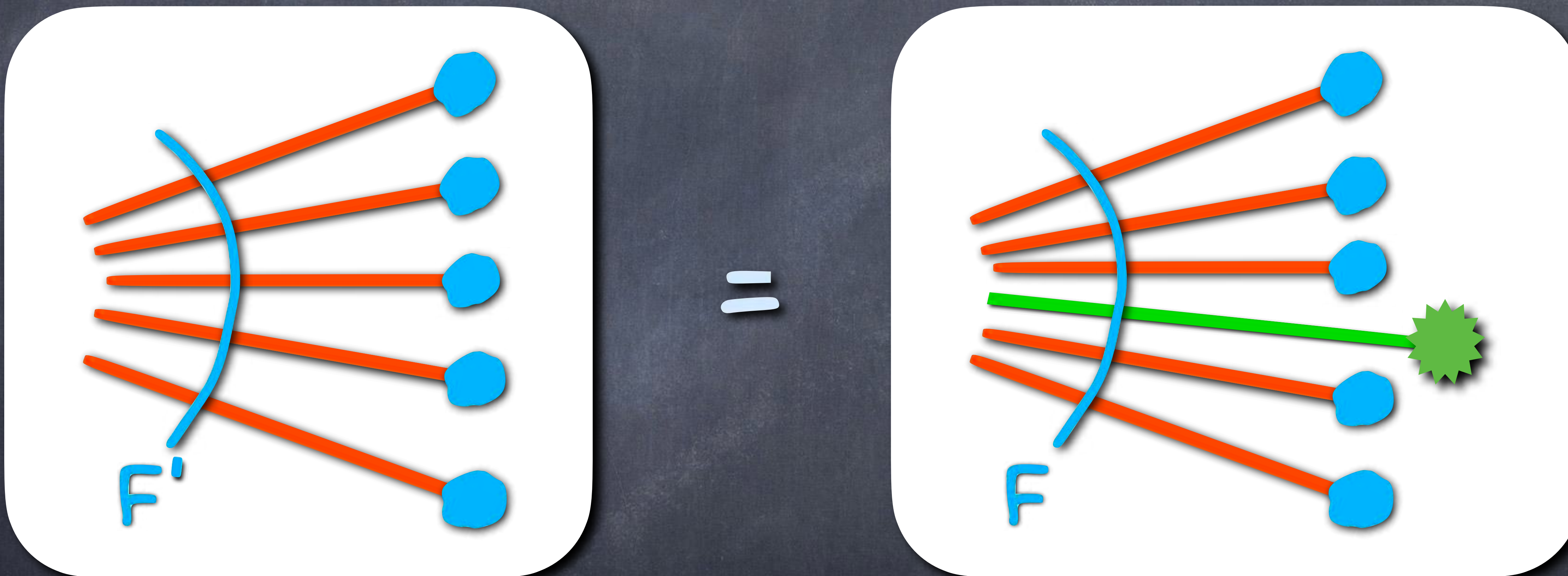
(F and G) - structure on A

$$(F\circ G)[A] := \sum_{\pi\in \mathrm{Part}[A]} F[\pi]\times\prod_{B\in\pi} G[B]$$



(F OF G)-STRUCTURE ON A

$$F'[A] := F[A + \{\ast\}]$$



Equality between Species

$$F = G$$

$\exists$ natural isomorphism

Equality between Species

$$F = G$$

$$\theta_A : F[A] \xrightarrow{\ \sim\ } G[A]$$

Natural Bijections

$$\begin{array}{ccccc}
A & & F[A] & \xrightarrow[\theta_A]{\sim} & G[A] \\
\varphi \downarrow \wr & & F[\varphi] \downarrow \wr & \circlearrowleft & \downarrow \wr\, G[\varphi] \\
B & & F[B] & \xrightarrow[\theta_B]{\sim} & G[B]
\end{array}$$

$$\mathbb{L} \overset{?}{=} \mathbb{S}$$

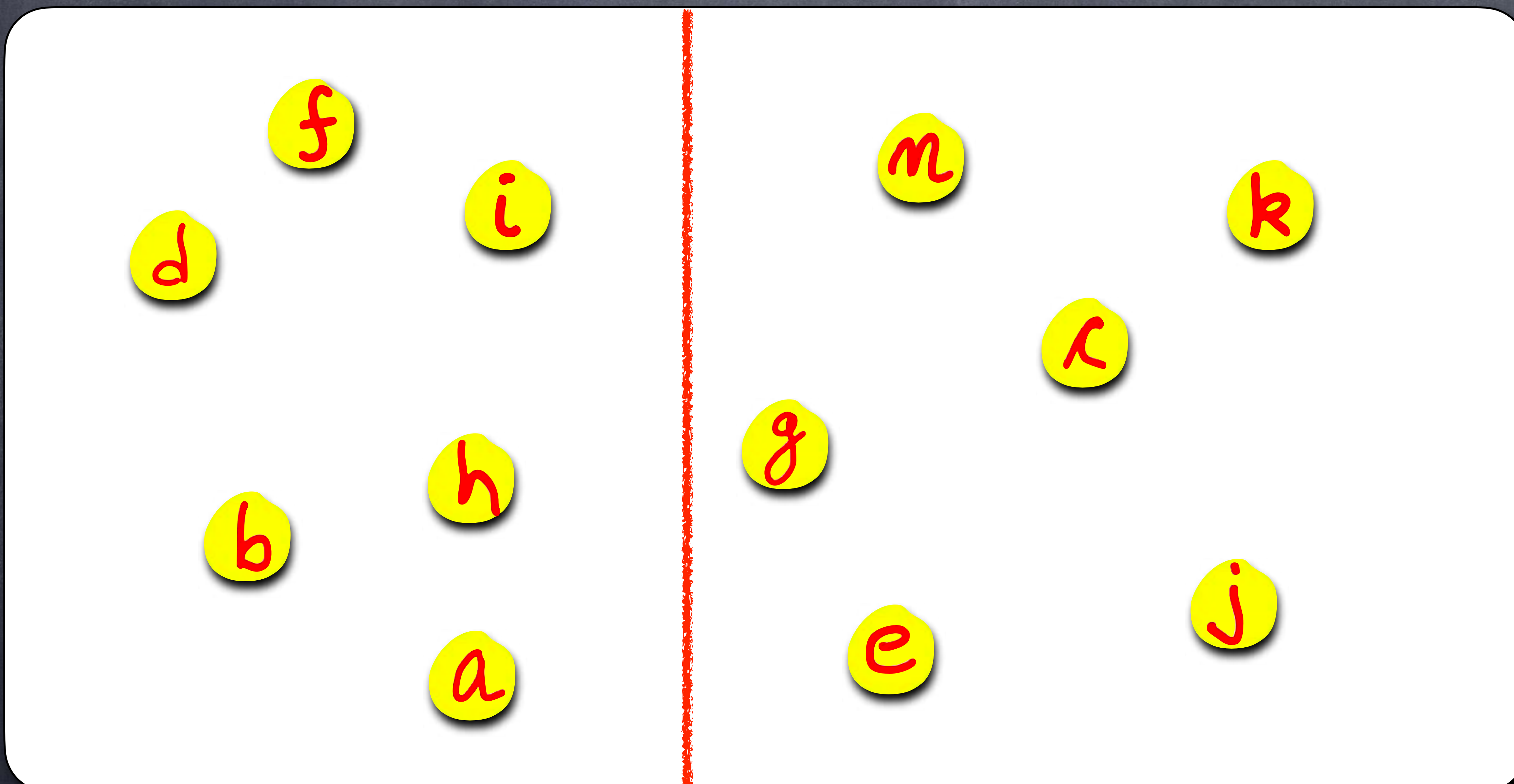
Lists

$(2, 5, 3, 1, 4)$

Permutations

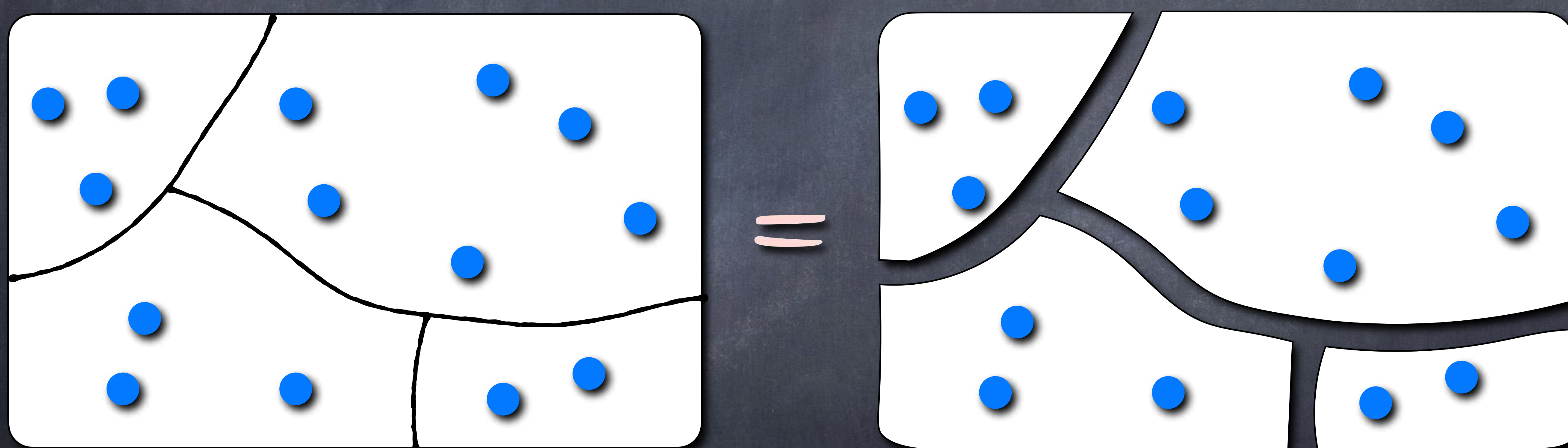
$\left\{\binom{1}{2}, \binom{2}{5}, \binom{3}{3}, \binom{4}{1}, \binom{5}{4}\right\}$

$\mathcal{P} = E \cdot E$



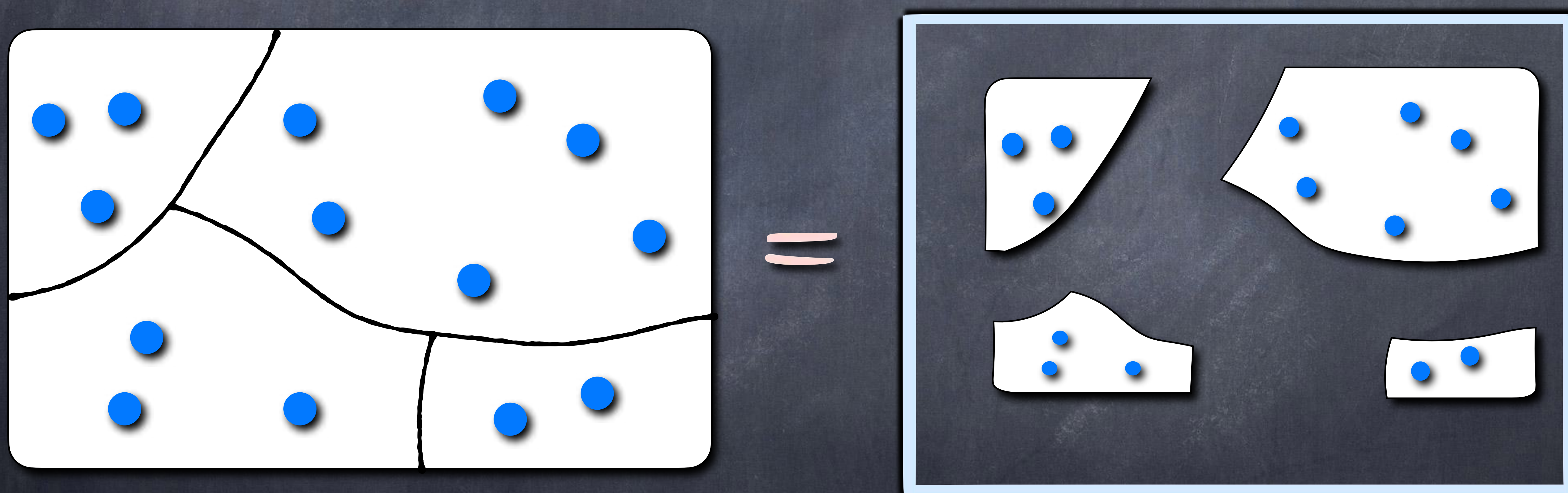
$\text{PART} = E(E_+)$

E_+ : NON EMPTY SETS

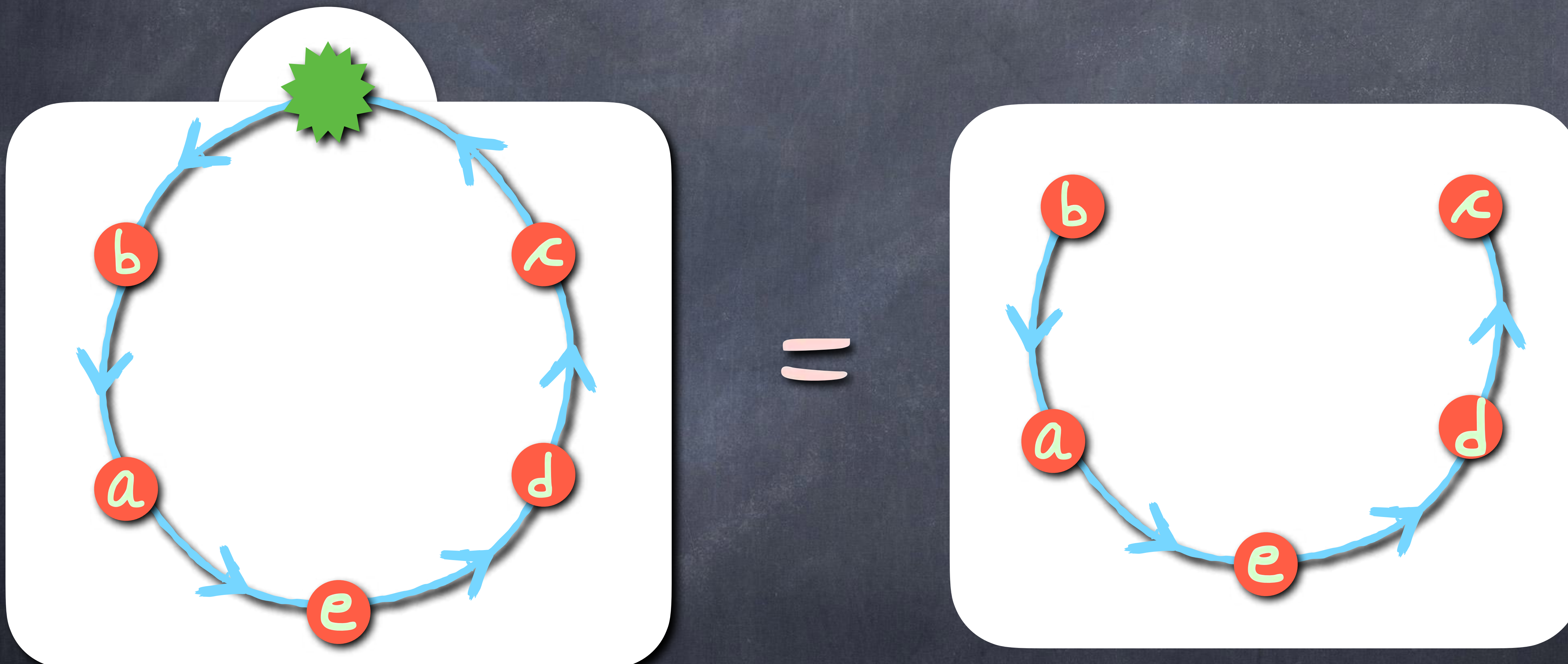


$\mathrm{Part} = E(E_+)$

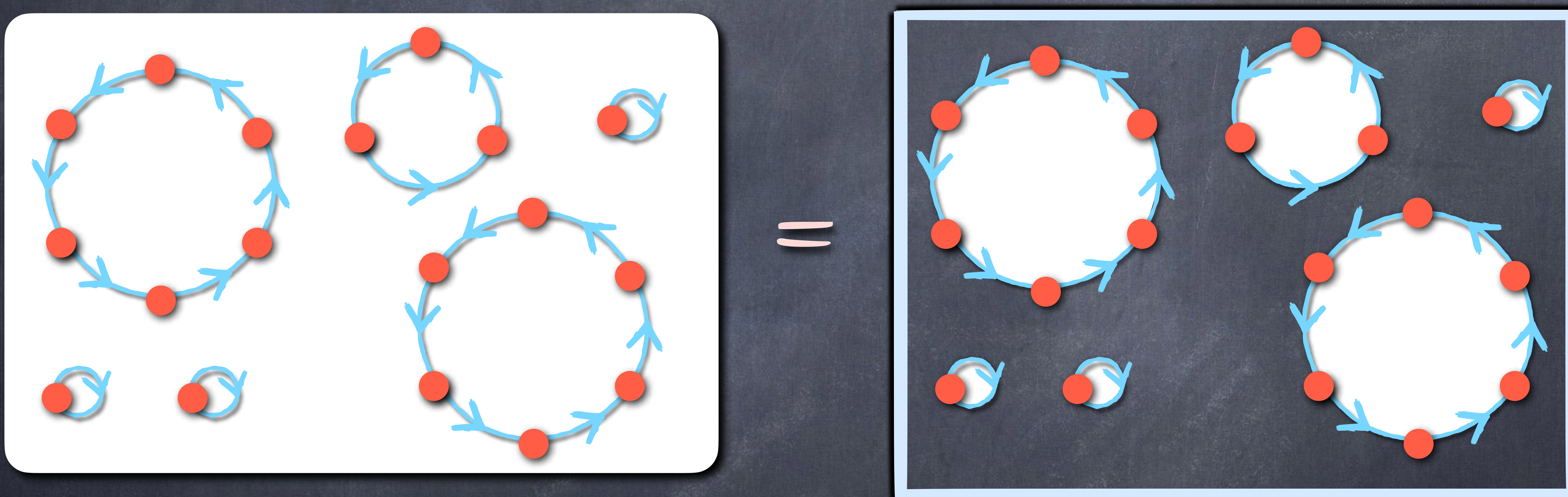
E_+ : Non empty sets



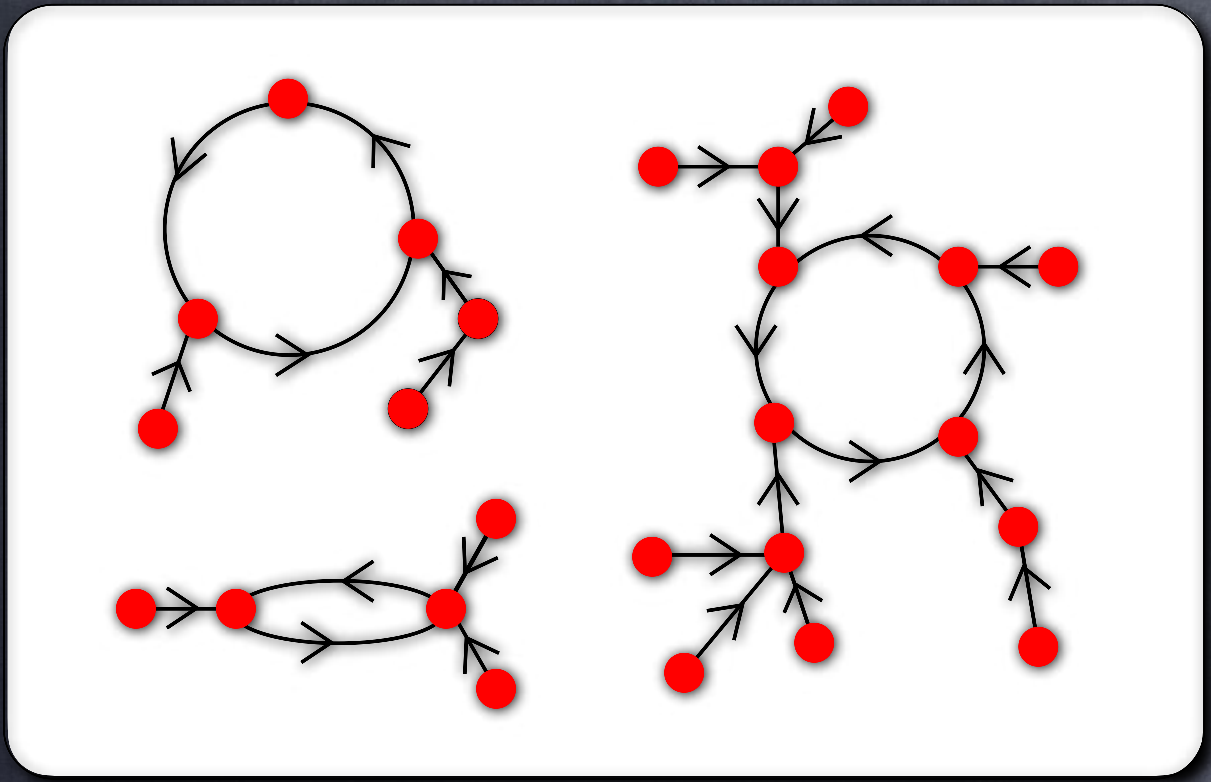
$$\mathcal{C}' = \mathbb{L}$$



$\mathfrak{S} = E(\mathfrak{C})$



End = $\mathfrak{S}(\mathcal{T})$



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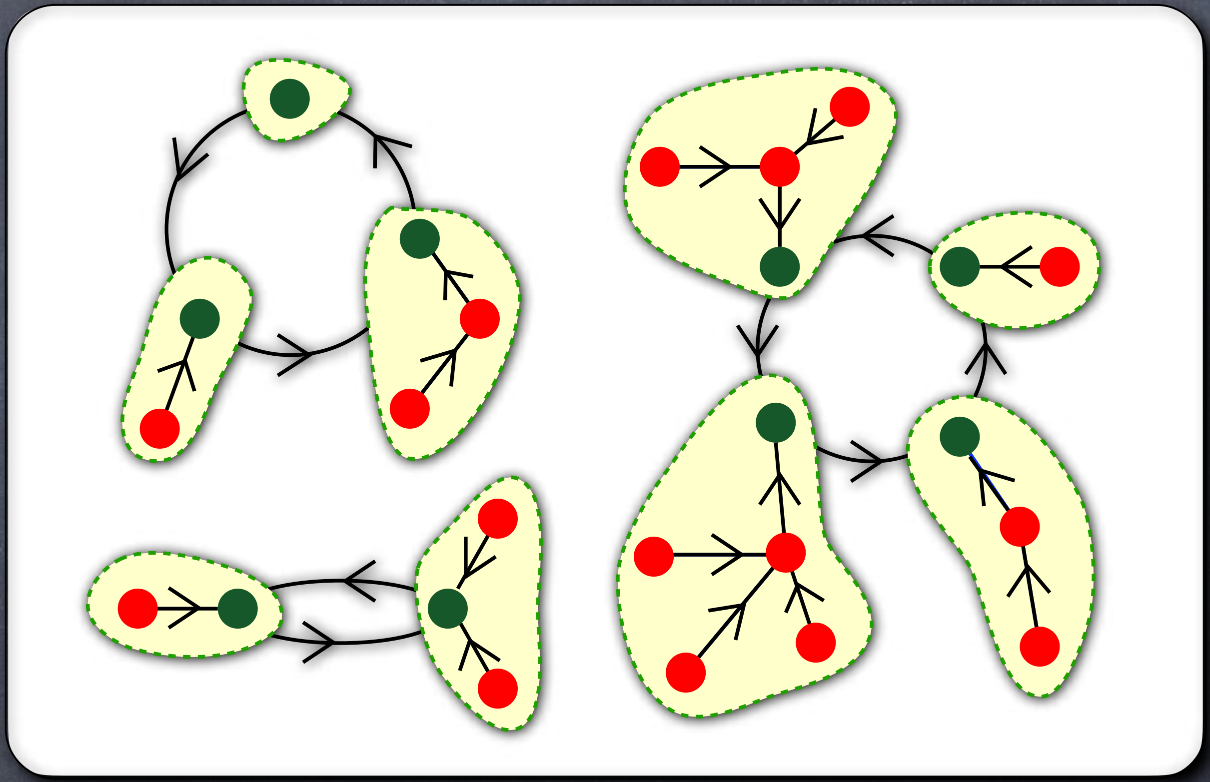
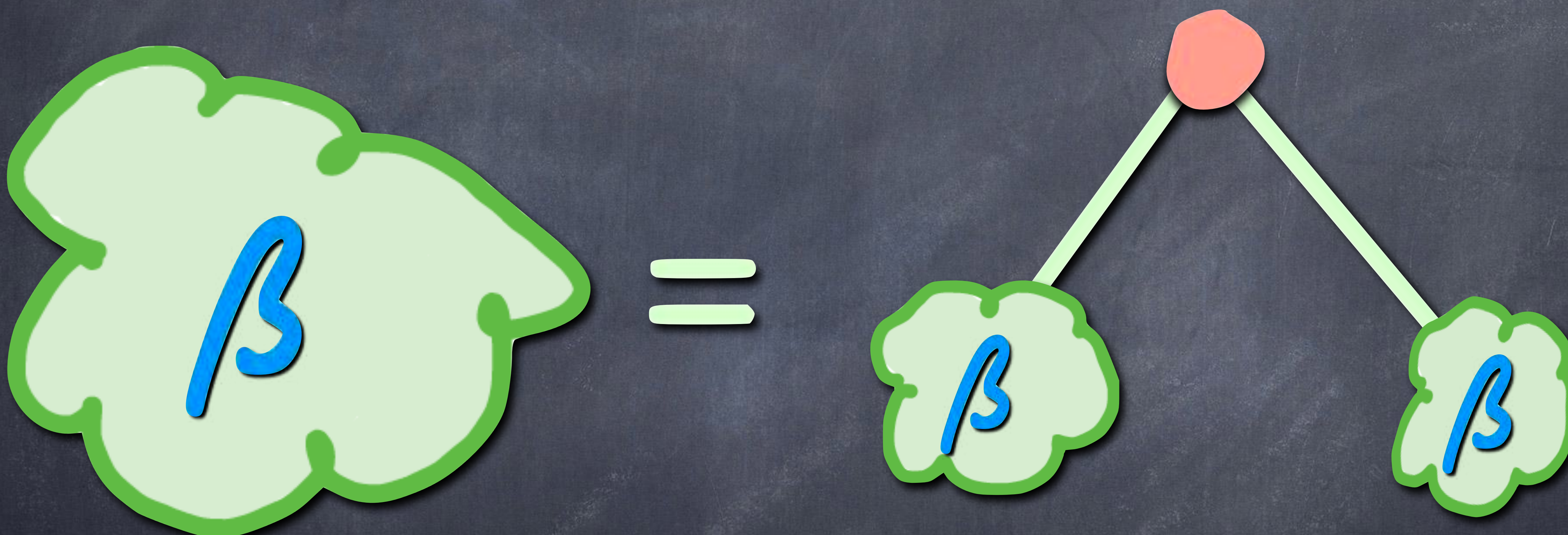


Implicit species theorem

$$\beta = 1 + X\beta^2$$



BINARY TREES

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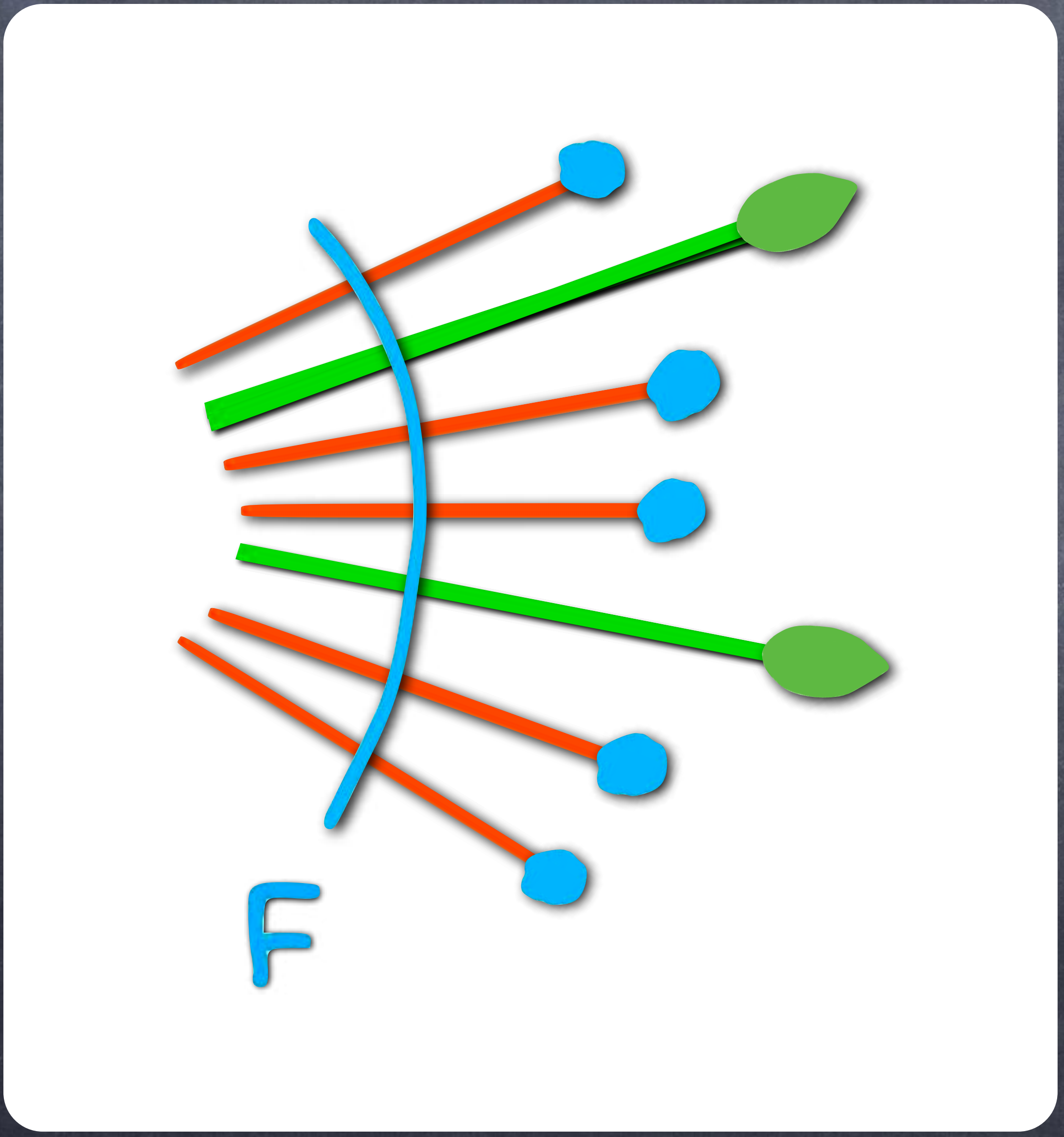
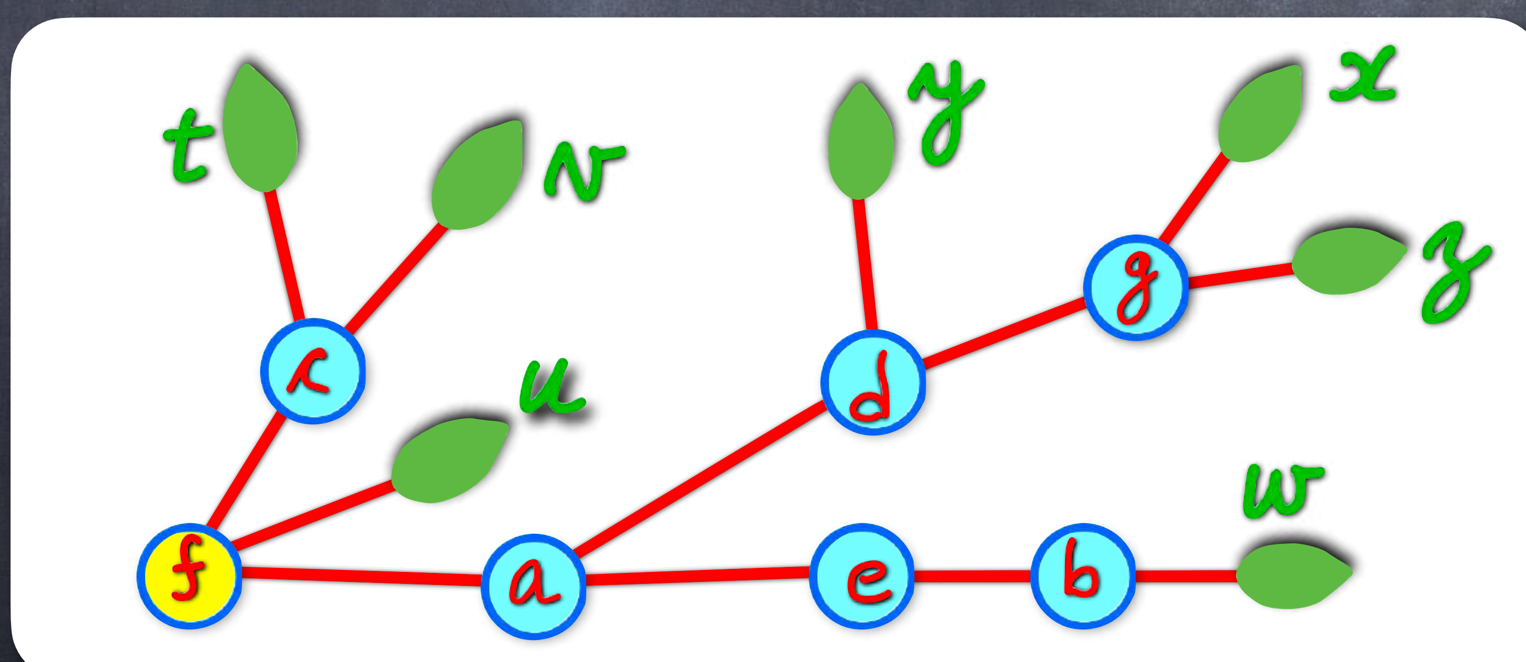
Species in Two Sorts



Species in Two Sorts

$$F[A,B] \qquad \varphi : A_1 \xrightarrow{\sim} A_2 \qquad \psi : B_1 \xrightarrow{\sim} B_2$$

$$F[\varphi,\psi] : F[A_1,B_1] \xrightarrow{\sim} F[A_2,B_2]$$



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Implicit species theorem

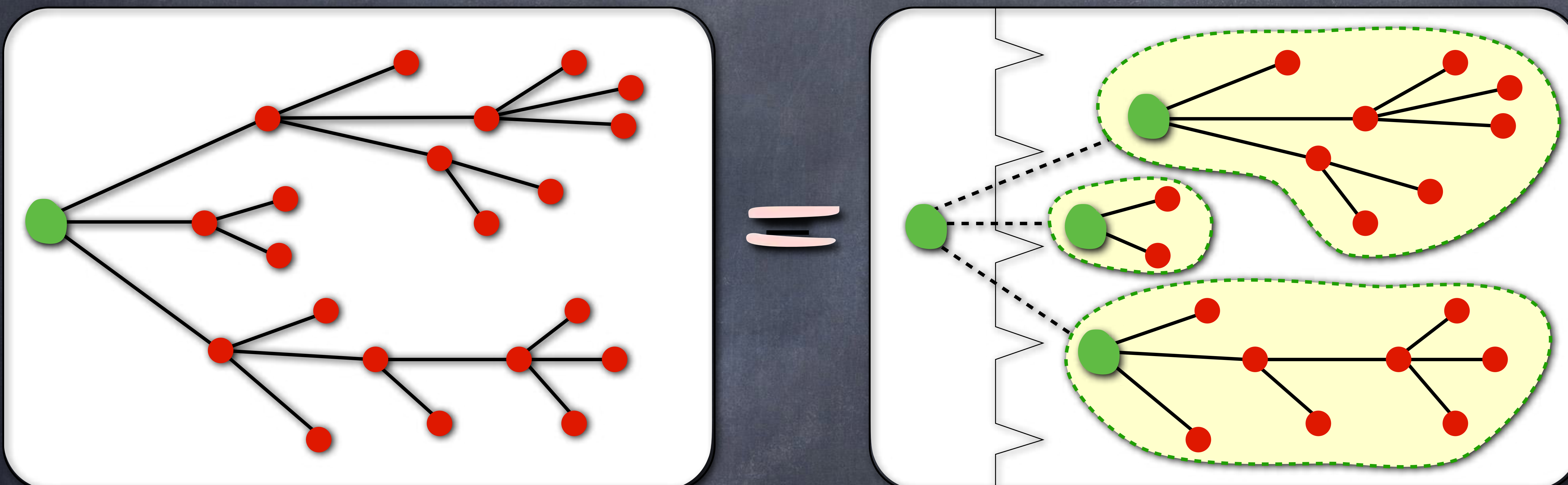
(Joyal 1981)

If $F[\emptyset,\emptyset]=\emptyset$ and $F[\emptyset,\{*\}]=\emptyset$

then there exists a unique species G (up to iso) such that

$$G = F(X, G) \quad \text{with} \quad G[\emptyset]=\emptyset$$

$$\Upsilon = X \cdot E(\Upsilon)$$



Variants

Functors

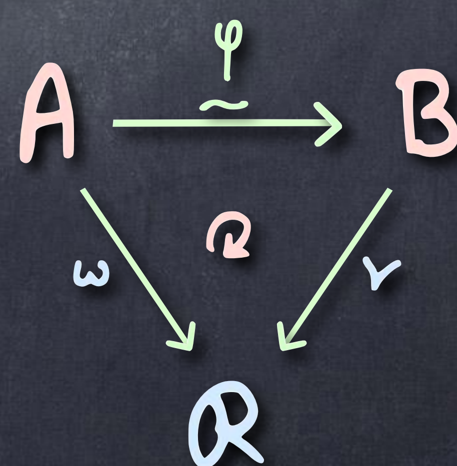
$$F : \mathbb{B}^k \longrightarrow \mathbb{B}^m$$

$$F : \mathbb{B} \longrightarrow \mathbb{B}_{\mathcal{R}}$$

$\mathcal{R}$ Ring

Objects

Arrows



Functors

$$F : \mathbb{L} \longrightarrow \mathbb{B}$$

$\mathbb{L}$: Category of finite ordered sets + increasing bijections

Functors

$F : \mathbb{B} \longrightarrow \mathbb{V}$

$\mathbb{V}$: Finite dimensional vector spaces (on $\mathbb{C}$) with bijective linear transformations

Operations in $\mathbb{V}$

1. $(F+G)[A] := F[A] \oplus G[A]$

2. $(F \cdot G)[A] := \bigoplus_{B+C=A} F[B] \otimes G[C]$

3. $(F \circ G)[A] := \bigoplus_{\pi \in \mathrm{Part}[A]} F[\pi] \otimes \bigotimes_{B \in \pi} G[B]$

FAMILIES OF GROUP ACTIONS

$$\mathbb{S}_A \times F[A] \longrightarrow F[A]$$

$$F : \mathbb{B} \longrightarrow \mathbb{B}$$

PERMUTATIONS

$$F : \mathbb{B} \longrightarrow \mathbb{V}$$

LINEAR

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Algebra Morphism

$\mathbb{B}$-Spe $\longrightarrow$ 

GENERATING SERIES

$$\mathbb{B}\text{-}\mathrm{Spe} \longrightarrow \mathbb{Q}[\![x]\!]$$

$$F(x) := \sum_{n \geq 0} f_n \frac{x^n}{n!}$$

$$f_n := \mathrm{card}(F[A]) \qquad \mathrm{card}(A) = n$$

GENERATING SERIES

$$\mathbb{B}\text{-}\mathrm{SPE} \longrightarrow \mathbb{Q}[\![x]\!]$$

$$F_\omega(x) := \sum_{n\geq 0} f_n^{(\omega)} \frac{x^n}{n!}$$

$$f_n^{(\omega)} := \sum_{s\in F[A]} \omega(s) \qquad \mathrm{CARD}(A)=n$$

ALGEBRA MORPHISM

1. $(F+G)(x) = F(x) + G(x)$
2. $(F \cdot G)(x) = F(x) \cdot G(x)$
3. $(F \circ G)(x) = F(G(x))$
4. $F'(x) = \frac{d}{dx} F(x)$

Generating series

- Singletons $X(x) = x$
- Sets $E(x) = e^x$
- Subsets $\mathcal{P}(x) = e^{2x}$
- Lists $\mathbb{L}(x) = 1/(1-x)$
- Permutations $\mathfrak{S}(x) = 1/(1-x)$
- Cycles $\mathcal{C}(x) = \log 1/(1-x)$

NOW THE FUN BEGINS

- BINARY TREES $\beta(x) = \dfrac{1 - (1-4x)^{1/2}}{2x}$
- ROOTED TREES $\Upsilon(x) := \sum_{n\geq 1} n^{n-1}\dfrac{x^n}{n!}$
- UNLABELLED STRUCTURES, PÓLYA THEORY, CHARACTERS, SYMMETRIC FUNCTIONS, ETC.

Version 10.6 Reference Manual

other version

Search

Home - Combinatorics

Combinatorial Species

This file defines the main classes for working with combinatorial species, operations on them, as well as some implementations of basic species required for other constructions.

This code is based on the work of Ralf Hemmecke and Martin Rubey's Aldor-Combinat, which can be found at http://www.risc.uni-linz.ac.at/people/hemmecke/aldor/combinat/index.html. In particular, the relevant section for this file can be found at http://www.risc.uni-linz.ac.at/people/hemmecke/AldorCombinat/combinatse8.html.

$$\Upsilon = X \cdot E(\Upsilon)$$

```
X = species.SingletonSpecies()
internal_node = species.SingletonSpecies(weight=q)
E = species.SetSpecies()
T = species.CombinatorialSpecies()
T.define(X*E(T))
```

```
T.cycle_index_series()
```

$$p_1 + p_{11} + \frac{3}{2}p_{111} + \frac{8}{3}p_{1111} + \frac{125}{24}p_{11111} + \frac{1}{2}p_{21} + p_{211} + \frac{9}{4}p_{2111} + \frac{5}{8}p_{221} + \frac{1}{3}p_{31} + \frac{2}{3}p_{311} + \frac{1}{4}p_{41}$$

END

$$F = G$$

$$\theta_A : F[A] \xrightarrow[\text{Natural Bijections}]{\sim} G[A]$$

Isomorphism of group actions

$$\mathbb{L} \neq \mathbb{S}$$

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