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Felix Klein and his Erlanger Programm

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Abstract. The present paper is dedicated to Felix Klein (1849–1925), one of the leading German mathematicians in the second half of the 19th century. It gives a brief account of his professional life. Some of his activities connected with the reform of mathematics teaching at German schools are mentioned as well. In the following text, we describe fundamental ideas of his Erlanger Programm in more detail. References containing selected papers relevant to this theme are attached.

Introduction

The Erlanger Programm plays an important role in the development of mathematics in the 19th century. It is the title of Klein's famous lecture *Vergleichende Betrachtungen über neuere geometrische Forschungen* [A Comparative Review of Recent Researches in Geometry] which was presented on the occasion of his admission as a professor at the Philosophical Faculty of the University of Erlangen in October 1872. In this lecture, Felix Klein elucidated the importance of the term *group* for the classification of various geometries and presented his unified way of looking at various geometries. Klein's basic idea is that each geometry can be characterized by a group of transformations which preserve elementary properties of the given geometry.

Professional Life of Felix Klein

Felix Klein was born on April 25, 1849 in Düsseldorf, Prussia. Having finished his study at the grammar school in Düsseldorf, he entered the University of Bonn in 1865 to study natural sciences. In 1866, he was offered an assistantship by the able mathematician and physicist Julius Plücker (1801–1868), who conceived the theory of line geometry. After passing his doctoral examination in 1868, Felix Klein consecutively visited Berlin, Paris and Göttingen. In 1870 in Paris, he struck up a friendship and a cooperation with Norwegian mathematician Sophus Lie (1842–1899). Both men understood the importance of the group concept in mathematics; Sophus Lie studied the theory of continuous transformation groups and Felix Klein studied discontinuous transformation groups from a geometric standpoint. At that time, fundamental ideas of his further work occurred to him.

When the Franco-Prussian war broke out in July 1870, Felix Klein returned to Germany and for a short time was employed in military service. In 1871, he started to lecture at the University of Göttingen. As early as 1872, at the age of only 23, Felix Klein was appointed full professor at the University of Erlangen. However, he stayed there only for three years. In 1875, he received an offer of the post at the Technische Hochschule in Munich and, consequently, moved there. In August 1875, he married Anne Hegel (1851–1927), a granddaughter of well-known German philosopher Georg Wilhelm Friedrich Hegel (1770–1831). During the period 1875–1880, Felix Klein published about seventy papers which covered group theory, theory of algebraic equations and function theory, all from a characteristically geometric viewpoint.

In 1880, Felix Klein was offered the new Chair of Geometry at the University of Leipzig. However, during the autumn 1882, he mentally collapsed and fell into a depression. His career as a top mathematician was over. He stayed in Leipzig until 1886 when he moved back to Göttingen. At the University of Göttingen he lectured on various parts of mathematics and physics. In 1913, he had to leave the University on grounds of his illness. During the First World War, he continued to give private lectures on mathematics at his home.

Felix Klein died on June 22, 1925 in Göttingen.

In mathematics, Felix Klein was interested not only in geometry but also in group theory, theory of algebraic equations and function theory. His merits are very universal. At the University of Göttingen he established a world-known mathematical centre and founded mathematical library as well. In 1876, Felix Klein became the chief editor of the mathematical journal *Mathematische Annalen* founded by Alfred Clebsch (1833–1872) and Carl Gottfried Neumann (1832–1925) in 1868. This journal specialized mainly in complex analysis, algebraic geometry and invariant theory. The reputation of the *Mathematische Annalen* began under Klein's leadership to surpass that one of the dominating *Journal für die reine und angewandte Mathematik* founded by August Leopold Crelle (1780–1855) although Crelle's *Journal*, edited by the Berlin mathematicians, was almost fifty years old by that time. Felix Klein took an active part in the multi-volume *Encyklopädie der mathematischen Wissenschaften mit Einschluß ihrer Anwendungen*, he personally edited the four volumes on mechanics. Amongst many other honours, Felix Klein had been a foreign member of the Royal Society of London for forty years and was awarded its highest honour, the Copley medal, in 1912. Next year, he became a member of the Berlin Academy of Sciences.

In the later years of his career, Felix Klein was also interested in the teaching of mathematics at German schools. He was fighting for its modernization and he made efforts for incorporation of the latest knowledge of mathematical science to classes at secondary schools and universities. With Klein's full support, the first Department of Mathematics Education was established at the University of Göttingen in 1886. The idea about additional education of mathematics teachers by means of lectures and holiday courses arose at that time. First courses under Klein's leadership took place in 1892. From his direct initiative, a programme of restructuring of subject matter at secondary schools was formulated in Merano in 1905. This programme called for the incorporation of function theory, infinitesimal calculus and groups of geometric transformations into the subject matter at grammar schools. In 1908, Felix Klein was appointed the President of the International Mathematics Instruction Commission. A great number of publications about the teaching of mathematics at all school grades was edited under his leadership.

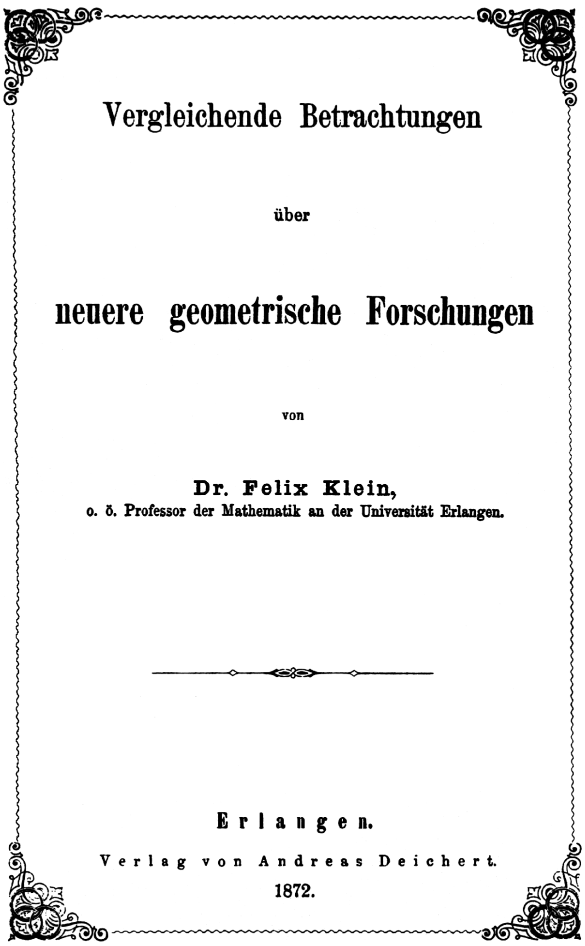
Figure 1. Felix Klein with his own signature.

Basic Idea of the Classification of Geometries

The essence of the classification of various geometries consists in the following. As is well known, Euclidean geometry considers the properties of figures that do not change under any motions; equal figures are defined as those that can be transferred onto one another by a motion. But instead of motions one may choose any other collection of geometric transformations and declare as equal those figures that are obtained from one another by transformations from this collection. This approach leads to another geometry which studies the properties of figures that are invariant under such transformations.

The relation between two figures must really be an equivalence; this means that it is a reflexive, symmetrical and transitive relation. It follows that the set of geometric transformations must be closed with respect to the composition of transformations, it must include the identity and the inverse of every transformation must be involved as well. In other words, the set of transformations must be a group.

The theory that studies the properties of figures preserved under all transformations of a given group is called the geometry of this group. The choice of distinct transformation groups leads to distinct geometries. Thus, the analysis of the group of motions leads to the common Euclidean geometry. When the motions are replaced by affine or projective transformations, the result is affine or projective geometry. Felix Klein proved in his work that starting from projective transformations that carry a certain circle or any other regular conic into itself, one comes to the non-Euclidean Lobachevski geometry.


Vergleichende Betrachtungen

über

neuere geometrische Forschungen

von

Dr. Felix Klein,
o. ö. Professor der Mathematik an der Universität Erlangen.

Erlangen.
Verlag von Andreas Deichert.
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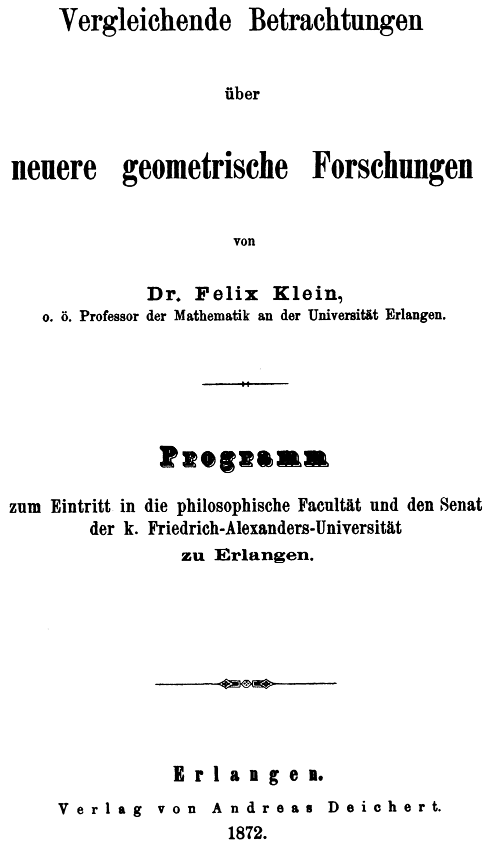
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Programm

zum Eintritt in die philosophische Facultät und den Senat
der k. Friedrich-Alexanders-Universität
zu Erlangen.

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Figure 2. The title page and the opening page of the Erlanger Programm.

The Erlanger Programm

As stated above, in October 1872, Felix Klein addressed at the University of Erlangen an inaugural lecture which has later become known and famous as the Erlanger Programm. In this lecture, the importance of the term *group* for the classification of various geometries is elucidated. The whole Programm consists of ten chapters. Fundamental ideas of Klein's classification of various geometries are presented in the first chapter where the following definition of such a geometry is stated:

> *Have a geometric space and some transformation group. A geometry is the study of those properties of the given geometric space that remain invariant under the transformations from this group. In other words, every geometry is the invariant theory of the given transformation group.*

Felix Klein emphasizes that the transformation group can be an arbitrary group.

This definition served to codify essentially all the existing geometries of the time and pointed out the way how to define new geometries as well. Until that time various types of geometry, e.g. Euclidean, projective, hyperbolic, elliptic and so on, were all treated separately. Felix Klein set forth in his Programm a unified conception of geometry that was far broader and more abstract than any one contemplated previously. At that time in Germany and elsewhere, much debate was going on about the validity of the recently developed non-Euclidean geometries. Felix Klein demonstrated in his Programm that they could be modelled in projective geometry associated with Euclidean geometry. Since no one doubted the validity of Euclidean geometry, this important insight served to validate non-Euclidean geometries as well.

In the second chapter of the Erlanger Programm, Felix Klein defines an ordering of geometries in such a way that he transfers the inclusion relation among various transformation groups to the corresponding geometries. Replacing some transformation group by other transformation group in which the original group is involved, only a part of the former geometric properties remains invariant. The passage to a larger group or a subgroup of a transformation group makes it possible to pass from one type of geometry to another one. In this way the Erlanger Programm codified a simple, but very important principle of ordering of particular geometries.

In order to illustrate Klein's fundamental ideas, let M be the set of all points of an ordinary plane and consider the set G of all geometric transformations of the set M consisting of translations, rotations, reflections and their products. Since the composition of any two such transformations and the inverse of any such transformation are also such transformations and the identity is involved in the set G, it follows that G is a transformation group. The resulting geometry is common plane Euclidean geometry, G is the isometry group. Since geometric properties such as length, area, congruence and similarity of figures, perpendicularity, parallelism, collinearity of points and concurrence of lines are invariant under the group G, these properties are studied in plane Euclidean geometry.

If, now, the group G is enlarged by including, together with all geometric transformations resulting from translations, rotations and reflections, the homothety transformations and all transformations composite from all above mentioned transformations, we obtain plane similarity geometry. Under this enlarged group, properties such as length, area and congruence of figures remain no longer invariant and hence are no longer subjects of the study in the framework of this geometry. However, similarity of figures, perpendicularity, parallelism, collinearity of points and concurrence of lines are still invariant and, consequently, constitute subject matter for the study of this geometry. Similarly, plane projective geometry is the study of those geometric properties which remain invariant under the group of the so-called projective transformations. Of the previously mentioned properties, only collinearity of points and concurrence of lines still remain invariant. An important invariant under this group of geometric transformations is the cross ratio of four collinear points as well.

In the table 1, there are seven basic geometric properties selected and for every from four chosen transformation groups there is shown whether given properties are invariant under such transformations or not.

Table 1. Some transformation groups and their invariants.

property	isometry group	similarity group	affine group	projective group
location	variable	variable	variable	variable
length	invariant	variable	variable	variable
area	invariant	variable	variable	variable
perpendicularity	invariant	invariant	variable	variable
parallelism	invariant	invariant	invariant	variable
collinearity	invariant	invariant	invariant	invariant
concurrence	invariant	invariant	invariant	invariant

Particular groups stated in the table 1 can be ordered by the inclusion relation in this way:

$$\begin{matrix}\text{isometry}\\ \text{group}\end{matrix} \subset \begin{matrix}\text{similarity}\\ \text{group}\end{matrix} \subset \begin{matrix}\text{affine}\\ \text{group}\end{matrix} \subset \begin{matrix}\text{projective}\\ \text{group}\end{matrix}$$

Every transformation group defines corresponding geometry. Thus, isometry group defines Euclidean geometry, similarity group defines similarity geometry, affine group defines affine geometry and projective group defines projective geometry. From the scheme above we obtain subsequent scheme which shows the relationship among principal geometries:

$$\begin{matrix}\text{Euclidean}\\ \text{geometry}\end{matrix} \supset \begin{matrix}\text{similarity}\\ \text{geometry}\end{matrix} \supset \begin{matrix}\text{affine}\\ \text{geometry}\end{matrix} \supset \begin{matrix}\text{projective}\\ \text{geometry}\end{matrix}$$

It is worth to stress that Felix Klein used in his work the latest knowledge of group theory and invariant theory of that time. Although in the present day the Erlanger Programm is considered as Klein's most important mathematical accomplishment, Klein's geometric results were neither immediately understood nor accepted at that time. Yet during the following twenty years, when it was available only as a booklet from Erlangen, it remained widely unknown. Later it was found out that during this period several other mathematicians, notably Henri Poincaré (1854–1912), arrived independently at similar ideas. The Erlanger Programm has become well-known not until it was reprinted in the journal *Mathematische Annalen* in 1893.

Conclusion

In the Erlanger Programm, Felix Klein set forth a unified conception of geometry that was far broader and more abstract than any one contemplated previously. It served to validate non-Euclidean geometries and define new geometries as well. Although it does not comprise some important branches of geometry, e.g. Riemannian geometry, it had a substantial stimulating effect on the subsequent development of geometry.

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