

THE q -FOULKES CONJECTURE

HISTORICAL BACKGROUND AND MATHEMATICAL BACKGROUND



DUDLEY ERNEST
LITTLEWOOD
(1903 - 1979)

PLETHYSM

POLYNOMIAL CONCOMITANTS AND INVARIANT MATRICES

J. LONDON MATH. SOC., 1936

A wider problem here suggests itself. Any invariant matrix of an invariant matrix must be an invariant matrix of the original matrix, and thus expressible as the direct sum of irreducible invariant matrices.

Thus

$$[A^{(\lambda)}]^{(\mu)} = \sum k_{\lambda\mu\nu} A^{(\nu)}.$$

Hence we may define a new type of multiplication of S -functions

$$\{\lambda\} \otimes \{\mu\} = \sum k_{\lambda\mu\nu} \{v\}.$$

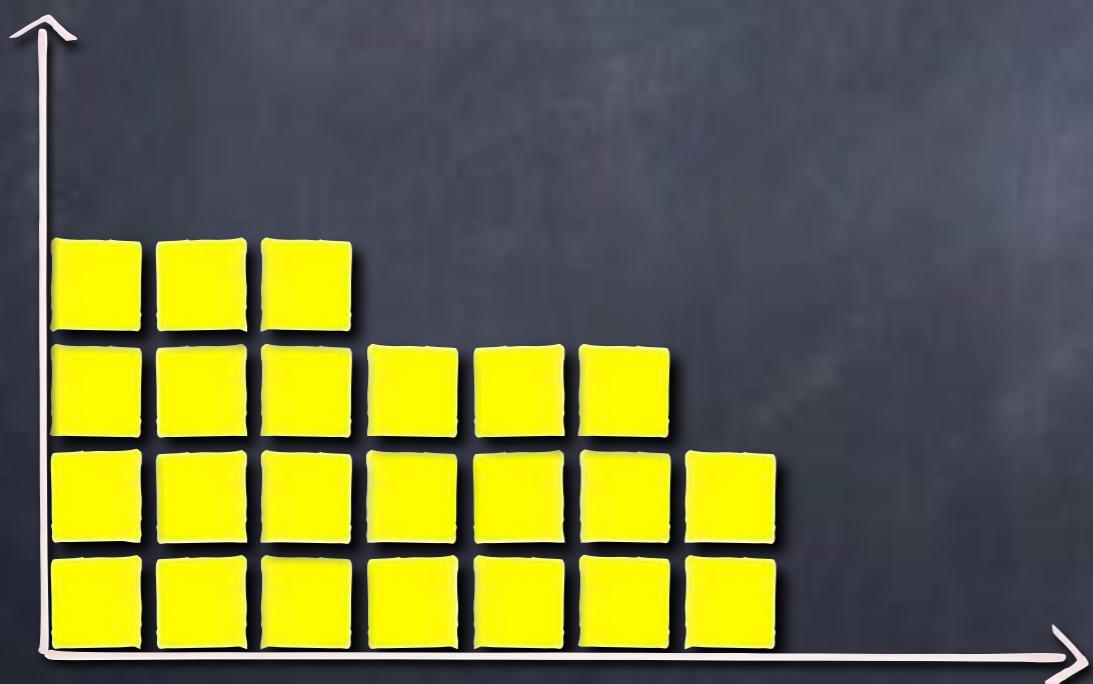
$$\{\lambda\} \otimes \{\mu\}$$

$$\sigma_\mu [\sigma_\lambda]$$

PLETHYSM

$$\{\lambda\} \otimes \{\mu\} = \Delta_\mu [\Delta_\lambda] = \Delta_\mu \circ \Delta_\lambda$$

PARTITIONS



$$\mu = \mu_1 \mu_2 \cdots \mu_k$$

$$\lambda = \lambda_1 \lambda_2 \cdots \lambda_j$$

POLYNOMIAL FUNCTORS

$$F : \mathcal{V}\text{ECT}_K \longrightarrow \mathcal{V}\text{ECT}_K$$
$$\text{GL}(V) \xrightarrow{F(-)} \text{GL}(F(V))$$

$S^k(V)$ IRREDUCIBLE
REPRESENTATION
OF $\text{GL}(V)$

CHARACTER

$$\mathcal{V} = \mathbb{K}\{N_1, N_2, \dots, N_n\}$$

$$\left. \begin{array}{l} T_X : \mathcal{V} \longrightarrow \mathcal{V} \\ T_X(n_i) := x_i N_i \end{array} \right\} \begin{array}{l} \text{ABSTRACT} \\ \text{DIAGONAL} \\ \text{MATRIX} \end{array} \quad x = x_1, x_2, \dots, x_n$$

$$F(x) := \text{TRACE}(F(T_X))$$

This is a SYMMETRIC function
of the x_i 's.

irreducible character

$$S^\lambda(x) = \Delta_\lambda(x)$$

← SCHUR FUNCTIONS

SCHUR Positivity

$$F = \bigoplus_{\lambda} a_{\lambda} S^{\lambda} \quad a_{\lambda} \in \mathbb{N}$$

$$F(x) = \sum_{\lambda} a_{\lambda} s_{\lambda}(x)$$

ALSO GRADED VERSION

$$a_{\lambda}(q) \in \mathbb{N}[q]$$

INTEGER COEFFICIENT
POLYNOMIAL

SCHUR Positivity

$f \leqslant_s g$ IFF

$f - g$ is SCHUR Positive

CLOSED UNDER ALL
USUAL OPERATIONS

SCHUR - POSITIVITY
IS RARE

AMONG
POSITIVE COEFFICIENT
HOMOGENEOUS DEGREE d
SYMMETRIC FUNCTIONS

$$d = 6$$

PROPORTION OF
SCHUR-POSITIVE

$\frac{1}{1027458432000}$

Functor Composition
=
RHYTHM OF CHARACTERS

$$(F \circ G)(x) = F[G(x)]$$

EXAMPLES

$$h_2[h_3] = \Delta_6 + \Delta_{42} \quad \Delta_r = \Delta_r(x)$$

$$h_2[h_4] = \Delta_8 + \Delta_{44} + \Delta_{62}$$

$$h_2[h_5] = \Delta_{10} + \Delta_{82} + \Delta_{64}$$

$$h_3[h_2] = \Delta_6 + \Delta_{42} + \Delta_{222}$$

$$h_4[h_2] = \Delta_8 + \Delta_{62} + \Delta_{44} + \Delta_{422} + \Delta_{2222}$$

EXAMPLES

$$\begin{aligned} h_4[h_3] = & \Delta_{12} + \Delta_{10,2} + \Delta_{93} \\ & + \Delta_{84} + \Delta_{822} + \Delta_{741} + \Delta_{732} \\ & + \Delta_{66} + \Delta_{642} + \Delta_{6222} \\ & + \Delta_{542} + \Delta_{444} \end{aligned}$$

$$h_1^a[h_1^b] = h_1^{ab}$$

Foulkes Conjecture



HERBERT OWEN
FOULKES
(1907 - 1977)

CONCOMITANTS OF THE QUINTIC AND SEXTIC UP TO DEGREE
FOUR IN THE COEFFICIENTS OF THE GROUND FORM

H. O. FOULKES*.

I. D. E. Littlewood (1) has shown that there is an exact correspondence between the S -functions appearing in the "new multiplication" $\{\lambda\} \otimes \{q\}$ and those concomitants, reducible or otherwise, of a ground form of type $\{\lambda\}$ which are of degree q in the coefficients of the ground form. He has also given several methods of computing such products and has obtained the number and types of concomitant for the cubic up to degree six in the coefficients, and for the quartic up to degree five in the coefficients†.

At the present time the ground form and its concomitants increase in complexity rapidly, so that it is difficult to obtain the complete list of concomitants for a given ground form.

J. OF LONDON MATH. SOC.
1950

A proof by
 S -functions of the general theorem underlying this assumption has not yet been obtained. The theorem is that for integers m, n , where $n > m$, the product $\{m\} \otimes \{n\}$ includes all terms of $\{n\} \otimes \{m\}$.

Two Formulations $a < b$

SYMMETRIC FUNCTION FORMULATION

$h_b[h_a] - h_a[h_b]$ is SCHUR POSITIVE

FACTORIAL FORMULATION

$$S^a \circ S^b \xrightarrow{\exists} S^b \circ S^a$$

$S^a(\gamma)$ SYMMETRIC POWER

Two Formulations $a < b$

Symmetric Function Formulation

$h_b[h_a] - h_a[h_b]$ is Schur Positive

Factorial Formulation

$$S^a \circ S^b \leftrightsquigarrow S^b \circ S^a$$

$S^a(\gamma)$ Symmetric Power



CHARLES HERMITE
(1822-1901)

SUR LA THEORIE DES FONCTIONS HOMOGENES À DEUX
INDETERMINEES.

Par M. HERMITE.

Mes premières recherches sur la théorie des formes à deux indéterminées, ont pour objet la démonstration de cette proposition arithmétique élémentaire, que les formes d'coefficients entiers et en nombre infini, qui ont les mêmes invariants, ne donnent qu'un nombre essentiellement limité de classes distinctes.

Section I.—Loi de Réciprocité.

Elle est contenue dans le théorème: A tout covariant d'une forme de degré m , et qui par rapport aux coefficients de cette forme est du degré p , correspond un covariant du degré m par rapport aux coefficients, d'une forme du degré p .

1854

$$h_b[h_a] = h_a[h_b]$$

$$x = x_1, x_2$$

A Proof

$$h_a \circ h_b [1+q] =$$

$$h_a [1+q+q^2+\dots+q^b] = \begin{bmatrix} a+b \\ a \end{bmatrix}_q$$

$$= h_b \circ h_a [1+q]$$



(a)

$$\alpha: S(S^i(\mathbb{C}^n)) \rightarrow \sum_{j \geq 0} S(\mathbb{C}^n \otimes \mathbb{C}^l)^{d_j}$$

(25a)

which consists of maps between each pair of homogeneous components

(b)

$$\alpha: S^p(S^i(\mathbb{C}^n)) \rightarrow S^i(S^p(\mathbb{C}^n)).$$

(25b)

When $n = 2$, it is not hard to see that the maps (25b) are all isomorphisms. This gives a very precise version of Hermite Reciprocity [7].

For $n > 2$, the maps (25b) cannot always be isomorphisms. In a conversation with the author, A. Garsia remarked that numerical evidence suggests that there should exist a GL_n -module embedding of $S^p(S^i(\mathbb{C}^n))$ into $S^i(S^p(\mathbb{C}^n))$ when $i \geq p$. This conjecture was also made in [4]. Thus perhaps it is reasonable to expect that the maps (25b) should be injective if $p \leq i$, and surjective if $i \leq p$.

ROGER EVANS HOWE

$$S^a \circ S^b \hookrightarrow S^b \circ S^a$$

$$S^a \circ S^b \longleftarrow S^b \circ S^a$$

NATURAL CANDIDATE

EXAMPLES

$$h_3[h_2] - h_2[h_3] = \Delta_{222}$$

$$h_4[h_2] - h_2[h_4] = \Delta_{422} + \Delta_{2222}$$

$$h_5[h_2] - h_2[h_5] = \Delta_{622} + \Delta_{442} + \Delta_{4222} + \Delta_{22222}$$

$$h_4[h_3] - h_3[h_4] = \Delta_{732} + \Delta_{6222} + \Delta_{5421}$$

MORE THAN 3 VARIABLES

$$\Delta_\lambda(x_1, x_2, \dots, x_m) = 0$$

WHEN $m < l(\lambda)$

EXAMPLES

$$h_5[h_3] - h_3[h_5] = \Delta_{10,3,2} + \Delta_{942} + \Delta_{9222} \\ + \Delta_{843} + \Delta_{8421} + \Delta_{8322} \\ + \Delta_{762} + \Delta_{7521} + \Delta_{743} \\ + \Delta_{7422} + \Delta_{72222} + \Delta_{6522} \\ + \Delta_{6441} + \Delta_{64221} + \Delta_{55311} \\ + \Delta_{5442}$$

SOME ADVANCES AND GENERALIZATIONS



ROBERT M. THRALL

ON SYMMETRIZED KRONECKER POWERS AND THE STRUCTURE
OF THE FREE LIE RING.*

By R. M. THRALL.

1. Introduction. This paper is divided into three chapters. In Chapter I foundations are laid for a general theory of representations of "power type" and their relationship with rings. Kronecker powers, symmetrized Kronecker powers, and the *Lie Representation* are introduced as transformations induced in certain modules of the free non-commutative ring, the free commutative ring, and the free Lie ring, respectively, by a class of ring automorphisms.

In Chapter II the starting point (§ 3) is a general discussion of a problem mentioned by Littlewood:¹ the analysis into irreducible invariant matrices of an invariant matrix of an invariant matrix. This is followed (§ 4) by more specific considerations in the case of the symmetrised Kronecker r -th power of a given invariant matrix. In § 5 formulas are obtained for the analysis of the symmetrised Kronecker r -th power of the symmetrised Kronecker m -th power for $r \leq 3$, all m , and for $m \leq 2$, all r . The chapter is concluded (§ 6) with a table giving the analysis of the symmetrised Kronecker r -th powers of the irreducible representations of the full linear group defined by partitions λ of m for all $x \in \mathfrak{sl}_n(\mathbb{C})$ with $\text{diag}(x) = 0$.

Foulkes Conjecture TRUE FOR $a=2$



MICHAEL BRION

THEOREM

$$\boxed{\begin{array}{c} \text{FUNCTORIAL FORMULATION} \\ S^a \circ S^b \leftarrow S^b \circ S^a \quad a \ll b \\ \hline \text{HOWE} \qquad \qquad \qquad \text{NATURAL CANDIDATE} \\ S^a(\gamma) \quad \text{SYMMETRIC POWER} \end{array}}$$

FOLKES CONJECTURE TRUE FOR



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Journal of Algebra 277 (2004) 579–614

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www.elsevier.com/locate/jalgebra



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Journal of Pure and Applied Algebra 139 (1999) 83–96

JOURNAL OF
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Generalized Foulkes' Conjecture and tableaux construction

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On Foulkes' conjecture

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$$a \leq c, d \leq b \quad ab = cd$$

$h_c[h_d] - h_a[h_b]$ is Schur Positive

A PROBLEM

FOR ALL

$$a \leq c, d \leq b \quad ab = cd$$

SHOW THAT

$$h_c \circ h_d [1 + q] - h_a \circ h_b [1 + q] \in \mathbb{N}[q]$$

A PROBLEM

FOR ALL

$$a \leq c, d \leq b \quad ab = cd$$

SHOW THAT

$$\left[\begin{smallmatrix} c+d \\ c \end{smallmatrix} \right]_q - \left[\begin{smallmatrix} a+b \\ a \end{smallmatrix} \right]_q \in \mathbb{N}[q]$$

TWO NEW EXTENSIONS

$$a \leq c, d \leq b \quad ab = cd \quad k$$

Conjecture 1

$$\boxed{h_a[h_b^k] \leq_d h_c[h_d^k]}$$

AND

$$\boxed{h_a[\Delta_{bb\cdots b}] \leq_d h_c[\Delta_{dd\cdots d}]}$$

A PROBLEM

SHOW THAT

$$h_a \circ h_b^k [1 + \frac{g}{f}] \leq_f h_c \circ h_d^k [1 + \frac{g}{f}]$$

FOR ALL

$$a \leq c, d \leq b \quad ab = cd \quad k$$

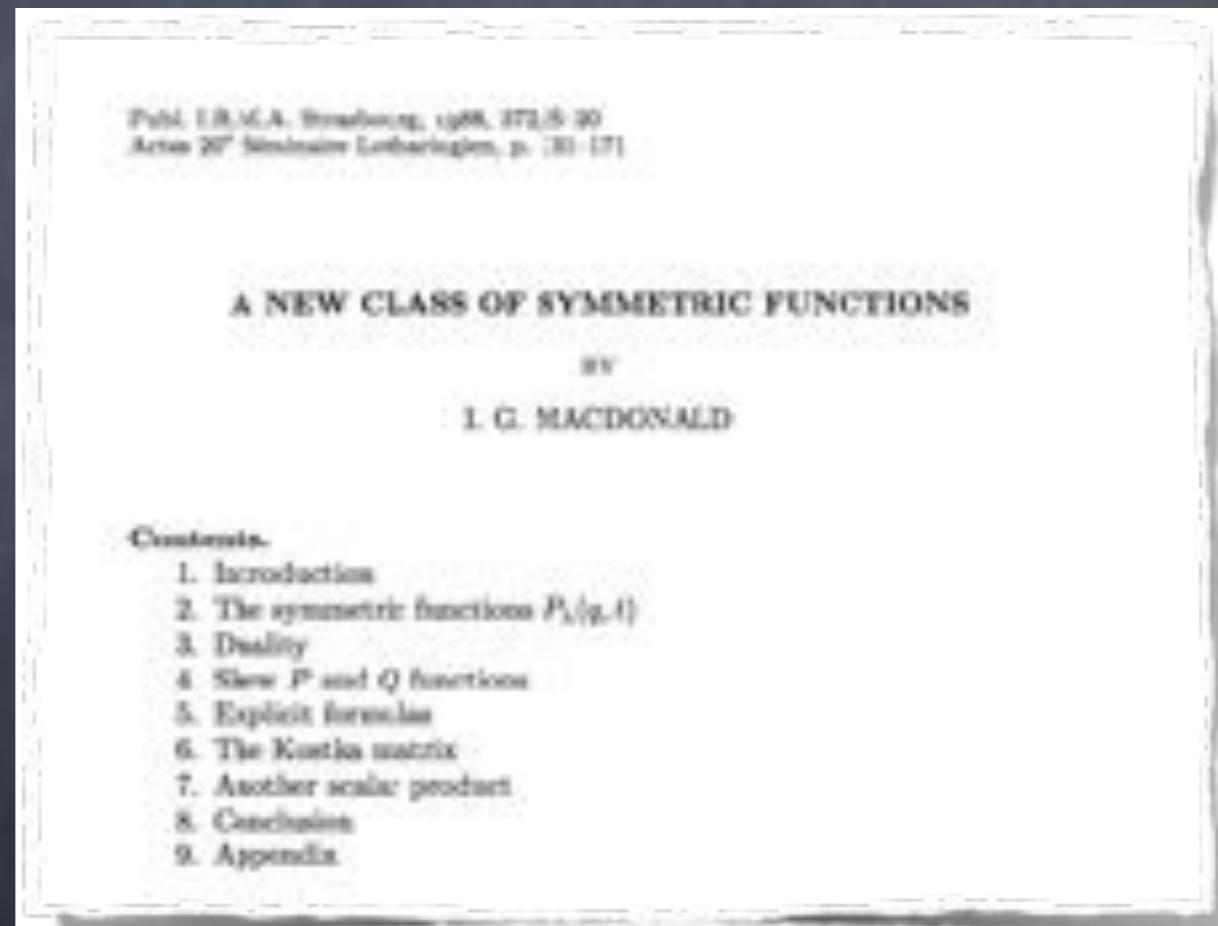
COMBINATORIAL MACDONALD POLYNOMIALS



IAN G. MACDONALD

COMBINATORIAL
MACDONALD
POLYNOMIALS

SCHUR POSITIVE



1988

EXAMPIES

$$H_2 = \Delta_2 + g \Delta_{\parallel}$$

$$H_3 = \Delta_3 + (g + g^2) \Delta_2 + g^3 \Delta_{\parallel}$$

EXAMPLES

$$\begin{aligned} H_4 = & \Delta_4 + (q + q^2 + q^3) \Delta_{31} \\ & + (q^2 + q^4) \Delta_{22} \\ & + (q^3 + q^4 + q^5) \Delta_{211} \\ & + q^6 \Delta_{1111} \end{aligned}$$

$$H_m(x; q, t) = H_m(x; q)$$

EXAMPLES

$$H_{22} = \Delta_4 + (qt + q + t)\Delta_{31} \\ + (q^2 + t^2)\Delta_{22} \\ + (q^2t + qt^2 + qt)\Delta_{21} \\ + q^2t^2\Delta_{111}$$

$$H_{31} = \Delta_4 + (q + q^2 + t)\Delta_{31} \\ + (q^2 + qt)\Delta_{22} \\ + (q^3 + qt^2 + qt)\Delta_{21} \\ + q^3t\Delta_{111}$$

SPECIALIZATIONS

$$H_\mu(x; 0, 0) = h_m(x)$$

$$H_\mu(x; 1, 1) = h_1(x)$$

$$H_\mu(x; 0, 1) = h_\mu(x)$$

$$H_\mu(x; 0, t) \Big|_{t^{\text{MAXDEG}}} = \sigma_\mu(x)$$

$$H_{\mu}(x; q, t) = H_{\mu_1}(x; t, q)$$

$$H_n(x; q) = \prod_{i=1}^n (1 - q^i) \cdot h_n\left[\frac{x}{1-q}\right]$$

$$H_{\mu_1\dots\mu_r}(x; t) = \prod_{i=1}^r (1 - t^i) \cdot h_n\left[\frac{x}{1-t}\right]$$

$$H_{\mu}(x; q, 1) = H_{\mu_1}(x; q) H_{\mu_2}(x; q) \cdots H_{\mu_r}(x; q)$$

$$H_{\mu}(x; q, t) = H_{\mu}(x; t, q)$$

$$H_n(x; q) = \prod_{i=1}^n (1 - q^i) \cdot h_n\left[\frac{x}{1-q}\right]$$

$$H_{||\dots|}(x; t) = \prod_{i=1}^n (1 - t^i) \cdot h_n\left[\frac{x}{1-t}\right]$$

$$H_{\mu}(x; q, 1) = \prod_{k \in \mu} H_k(x; q)$$

$$H_{\mu}(x; q, t) = H_{\mu'}(x; t, q)$$

$$H_n(x; q) = \prod_{i=1}^n (1 - q^i) \cdot h_n\left[\frac{x}{1-q}\right]$$

$$H_{\mu'}(x; t) = \prod_{i=1}^n (1 - t^i) \cdot h_n\left[\frac{x}{1-t}\right]$$

$$H_{\mu}(x; q, 1) = \prod_{k \in \mu} H_k(x; q)$$

$$H_{\mu'}(x; 1, t) = \prod_{k \in \mu'} H_k(x; t)$$

$$H_n(X; \mathbb{Q})$$

GRADED CHARACTER OF THE

- COINVARIANT RING OF S_n
- COHOMOLOGY RING OF THE
FULL FLAG MANIFOLD
- MODULE OF S_n -HARMONIC POLYNOMIALS

$H_\mu(x; o, t)$

GRADED CHARACTER OF
THE COHOMOLOGY RING \mathcal{H}_μ^*
OF PARTIAL FLAG VARIETY

$$h_a \circ H_{bb \dots b}(x; o, t) \in \text{Sym}^a(\mathcal{H}_{bb \dots b}^*)$$

THE q -FOULKES CONJECTURE AND (q,t) ANALOGS

Conjecture 2

$a < b$

$$\left[\frac{H_b[H_a] - H_a[H_b]}{1-f} \text{ is Schur Positive} \right]$$

$$\downarrow f = 0$$

$$\left[h_b[h_a] - h_a[h_b] \text{ is Schur Positive} \right]$$

$$\mathbb{F}_{a,b}(x;g) := \frac{H_b[H_a] - H_a[H_b]}{1-g}$$

$H_b[H_a] - H_a[H_b]$ is divisible by $1-g$

$$\downarrow g=1$$

$$h_i^b[h_i^a] - h_i^a[h_i^b] = 0 \quad \text{AT } g=1$$

$$\mathcal{F}_{a,b}(x; q) := \frac{H_b[H_a] - H_a[H_b]}{1 - q}$$

$$\begin{aligned}
\mathcal{J}_{23}(x; q) = & q^2 (q+1)^2 \Delta_{33} + q (q^2+1) (q+1)^2 \Delta_{321} \\
& + q^2 (q+1)^2 \Delta_{311} + (q+1) (q^2+1) \Delta_{222} \\
& + q (q+1) (q^2+1) (q^2+q+1) \Delta_{2211} \\
& + q^2 (q+1) (2q^2+q+1) \Delta_{21111} \\
& + q^3 (q+1) (q^2+1) \Delta_{111111}
\end{aligned}$$

$$\mathfrak{F}_{a,b}(x;0) = h_b[h_a] - h_a[h_b]$$

$$\mathfrak{F}_{2,3}(x;0) = \Delta_{222}$$

EXPANDING AS A POLYNOMIAL IN g

$$\begin{aligned}
 J_{a,b} &= (h_b[h_a] - h_a[h_b]) \\
 &+ g h_i \underbrace{(h_{b-1}[h_a] \cdot h_{a-i} - h_{a-1}[h_b] \cdot h_{b-1})}_{h_i h_i^\top (h_b[h_a] - h_a[h_b])} + \dots
 \end{aligned}$$

FOLKES \Rightarrow THIS IS SCHUR
POSITIVE

EXPANDING AS A POLYNOMIAL IN g

$$\mathcal{J}_{a,b} = (h_b[h_a] - h_a[h_b]) + g h_i \underbrace{(h_{b-1}[h_a] \cdot h_{a-1} - h_{a-1}[h_b] \cdot h_{b-1})}_{h_i h_i^\perp (h_b[h_a] - h_a[h_b])} + \dots$$

$$h_i h_i^\perp (h_b[h_a] - h_a[h_b])$$

FOLKES \Rightarrow THIS IS SCHUR
POSITIVE

h_i^\perp DUAL OF MULTIPLICATION
By h_i

CONJECTURE 3

$$a \leq c, d \leq b \quad ab = cd$$

$$\frac{H_c[H_d] - H_a[H_b]}{1-q} \text{ is SCHUR Positive}$$

$$\downarrow q = 0$$

$$h_c[h_d] - h_a[h_b] \text{ is SCHUR Positive}$$

Conjecture 4

$$a \leq c, d \leq b \quad ab = cd \quad k$$

$$\frac{H_c \circ H_{dd\cdots d} - H_a \circ H_{bb\cdots b}}{1-f}$$

is (f, t) - Schur Positive

Supporting Evidence

- # • COMPUTER ALGEBRA CALCULATIONS

$\text{P}(B \mid A) = \frac{\text{P}(A \cap B)}{\text{P}(A)} = \frac{\text{P}(B \mid A) \cdot \text{P}(A)}{\text{P}(A)} = \frac{\text{P}(B \mid A) \cdot \text{P}(A) + \text{P}(B \mid \bar{A}) \cdot \text{P}(\bar{A})}{\text{P}(A) + \text{P}(\bar{A})}$

MY MATHEMATICAL TELESCOPE

- $q = 1$, THE (k, b, κ, d) -
VERSION OF FOULKES
CONJECTURE HOLDS.

$$\mathfrak{F}_{a,b}(x;g) := \frac{H_b[H_a] - H_a[H_b]}{1-g}$$

$$\begin{aligned}
\mathfrak{J}_{23}(x, 1) &= 4 \Delta_{33} + 8 \Delta_{321} \\
&\quad + 4 \Delta_{3111} + 4 \Delta_{222} \\
&\quad + 12 \Delta_{2211} + 8 \Delta_{21111} + 4 \Delta_{111111} \\
&= 4 e_2^3
\end{aligned}$$

EXAMPLES

$$\mathcal{F}_{23}(x,1) = 4 e_2^3$$

$$\boxed{\mathcal{F}_{a,b}(x,1) \in \mathbb{N}[e_1, e_2, h_2]}$$

$$\mathcal{F}_{24}(x,1) = 8 e_2^4 + 16 e_2^3 h_2 \quad a < b$$

$$\mathcal{F}_{34}(x,1) = 24 e_1^4 e_2^3 h_2$$

$$\mathcal{F}_{25}(x,1) = 16 e_2^5 + 40 e_2^4 h_2 + 40 e_2^3 h_2^2$$

$$\mathcal{F}_{45}(x,1) = 16 e_1^{10} e_2^5 + 80 e_1^{10} e_2^3 h_2^2$$

$$\mathcal{F}_{56}(x,1) = 120 e_1^{10} e_2^5 h_2 + 200 e_1^{10} e_2^3 h_2^3$$

THEOREM

WE HAVE

$$\boxed{f_{a,b}(x,1) \in \mathbb{N}[e_1, e_2, h_2]} \quad a < b$$

EXPLICITLY GIVEN BY THE

FORMULA :

$$f_{a,b}(x,1) = \frac{e_1^{(a-2)b}}{2} \left(ab(b-a) e_1^{2(b-1)} e_2 + \binom{a}{2} (P^b - Q^b) - \binom{b}{2} (P^a - Q^a) \right)$$

WHERE

$$P := h_2 + e_2$$

$$Q := h_2 - e_2$$

THEOREM

$$a \leq c, d \leq b \quad ab = cd = n$$

$$\lim_{\epsilon \rightarrow 1} \frac{H_c[h_d] - H_c[h_b]}{1-\epsilon} = \frac{1}{2} \left(n(b-d) e_1^{n-2} e_2 + \binom{a}{2} e_1^{n-2b} P^b - \binom{c}{2} e_1^{n-2d} Q^d \right)$$

WHERE

$$P := h_2 + e_2 \quad Q := h_2 - e_2$$

\mathfrak{g} -STABILITY

$\bar{\mu}$: REMOVE LARGEST
PART FROM μ

$$\overline{\sigma}_\mu := \sigma_{\bar{\mu}}$$

$$\overline{43221} = 3221$$

Conjecture 5

$\overline{\mathfrak{f}}_{a,b+1} - \overline{\mathfrak{f}}_{a,b}$ is SCHUR POSITIVE
AND COEFFICIENTS STABILIZE

THEOREM



MICHEL BRION

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Stable properties of plethysm : on two conjectures of Foulkes

Michel BRION

Two conjectures made by H.O. Foulkes in 1950 can be stated as follows.
1) Denote by V a finite-dimensional complex vector space, and by $S_m V$ its m -th symmetric power. Then the $GL(V)$ -module $S_n(S_m V)$ contains the $GL(V)$ -module $S_n(S_p V)$ for $n > m$.
2) For any (decreasing) partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$, denote by $S_\lambda V$ the associated simple, polynomial $GL(V)$ -module. Then the multiplicity of $S_{(\lambda_1+p, \lambda_2, \lambda_3, \dots)} V$ in the $GL(V)$ -module $S_n(S_{m+p} V)$ is an increasing function of p . We show that Foulkes' first conjecture holds for n large enough with respect to m (Corollary 1.3). Moreover, we state and prove two broad generalizations of Foulkes' second conjecture. They hold in the framework of representations of connected reductive groups, and they lead e.g. to a general analog of Hermite's reciprocity law (Corollary 1 in 3.3).

$$f = 0$$

THEOREM

$\hat{f} = 1$

$$\tilde{J}_{a,b+1}(x,1) = e_i^a \tilde{J}_{a,b}(x,1) + \Delta_{a,b}$$

WHERE $\Delta_{a,b}$ IS SEMI POSITIVE.

THIS IMPLIES STABILITY AT $\hat{f} = 1$.

FIN