

Algebraic Combinatorics and Coinvariant Spaces



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Introduction

Whenever one encounters the number $n!$ playing a natural role in some mathematical context, it is almost certain that some interesting combinatorial objects are lurking around. This is going to be a recurrent theme in this book, with a mostly algebraic upbeat. Up to very recently a conjecture, known simply as the $n!$ **conjecture**, was still open. Although now settled, many important questions surrounding this conjecture continue to remain mysterious. The actual statement¹ of the conjecture made it appear deceptively easy. Without going into details, it simply stated that the dimension of a certain space of n variables polynomials had to be equal to $n!$. The only known proof of this (found after at least a decade of intense research by many top level mathematicians) is rather intricate and makes use of Algebraic Geometry notions that lie beyond the intended scope of this book. This is but one instance of the manifold interesting algebraic incarnations of $n!$. If one expands such consideration to families of integers or polynomials closely related to $n!$, then the richness of the algebraic landscape becomes truly impressive. On a tour of this landscape, one could come across notions such as the Cohomology Rings of Flag Varieties, the Double Affine Hecke Algebras of Cherednik, Hilbert Schemes, Inverse Systems, Gromov-Witten Invariants, etc. However it would be a very tall order to present all of these ties in a book that is intended to be short and accessible. In fact, most of these subjects are not addressed here, and the emphasis is rather on the Invariant Theory and Finite Group Representation Theory side of the story. This bias is sure to hide much of the beauty and unity of the material considered, and it surely makes it a bit more mysterious than it should be. We could probably have painted a clearer and crisper picture using notions of reductive algebraic groups, but this remained in the to-do-after-more-thought pile. Anyway, when trying to understand a deep mathematical subject, it is often only with hindsight that one finally understands how simple and clear everything should have been right from the start.

In the last 25 years, there has been a fundamental transformation and expansion of the scope and depth of Combinatorics. A good portion of this evolution has given rise to an independent subject that has come to be known as Algebraic Combinatorics, and whose goal is to study various deep interactions between Combinatorics, Representation Theory, Algebraic Geometry, and other classical subfields of Algebra. One of the nice and rich feature of these interactions is certainly a renewed interest in the combinatorics of symmetric polynomials, or more generally of invariant

¹See Section 10.1 for the actual formulation.

polynomials for finite group of matrices. The origin of this trend can at least be traced back to a 1979 seminal paper of Richard Stanley, on *Invariants of Finite Groups and their Applications to Combinatorics* [?]. This paper helps pinpoint a period of joint and complementary efforts involving a large group of researchers among which one should certainly mention Björner, Lascoux, Garsia, Schutzenberger, and Rota, as being major original players on the combinatorial side of this story.

Another natural line of inquiry leads to the study of quotients associated to the diagram of inclusions of algebras which appear in Figure (1). Most of the spaces at the bottom of this diagram will at least be mentioned in our discussion, and all arrows have a significant role. The top part corresponds to a non-commutative analog of the bottom part, but this top part did not find its way into the final version of this book. Still it is interesting to bear the whole diagram in mind, both as a background map for our exploration and as a map for further work. Many aspect are missing in this picture, most notably are extensions along a fourth dimensional axis parametrized by the choice of the underlying group action. Along these lines the prototypal object

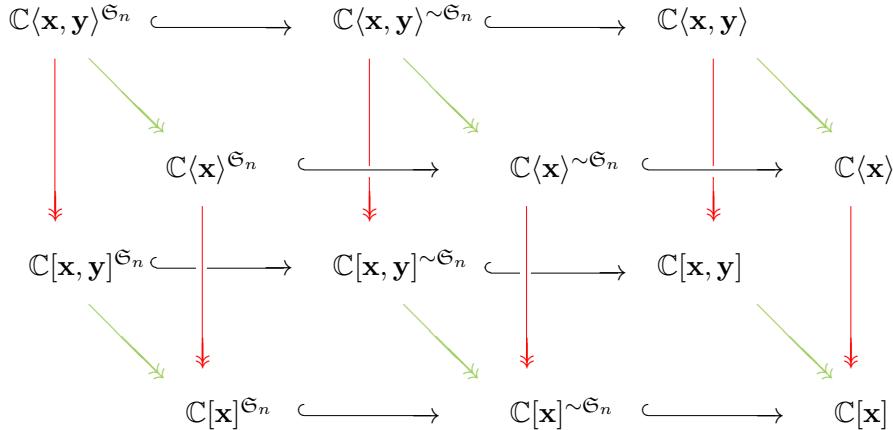


Figure 1: General overview.

is certainly the S_n -module $\mathbb{C}[\mathbf{x}]/\langle\mathbb{C}[\mathbf{x}]_+^{S_n}\rangle$, where $\langle\mathbb{C}[\mathbf{x}]_+^{S_n}\rangle$ denotes the ideal of $\mathbb{C}[\mathbf{x}]$ generated by constant-term-free symmetric polynomials. This quotient, often encountered in association with the ring $\mathbb{C}[\mathbf{x}]^{S_n}$ of S_n -invariant (symmetric) polynomials, plays an important role at least in the fields of Invariant Theory [?], Galois Theory [?], and Algebraic Geometry [?]. In the first of these contexts, it is known as the “coinvariant space” of the symmetric group; and in the last, it appears as the cohomology ring of the flag variety. It has finite dimension equal to $n!$, and it is in fact isomorphic to the left regular representation of S_n . This is but the symmetric group case of a more general story concerning finite reflection groups (see [?]). Recent renewed interest in these coinvariant spaces is closely tied with the study of the bigraded S_n -module of “diagonal coinvariant space” (see [?]). In turn these S_n -module have been considered in relation with the study of a remarkable family of two parameter symmetric functions known as “Macdonald functions”. Not only do Macdonald functions unify most of the important families of symmetric functions, but they

also play an interesting role in Representation Theory. It may also be worth mentioning that they appear naturally in the study of Calogero-Sutherland quantum many body systems in Statistical Physics (see [?]). On another note, the study of Macdonald functions has prompted the introduction of new families of symmetric functions such as the k -Schur functions (see [?]) which relate to the study of 3-point Gromov-Witten invariants as well as to WZW Conformal Field Theory.

To make this presentation short enough, and keep it lively, I have chosen to skip many proofs that can easily be found in the literature, especially when the presentation can easily be followed without explicitly writing down the relevant proof. In these instances a specific reference is given for the interested reader. However, some proofs were kept either because they teach something new about the underlying situation, or simply because they are very nice. This is a difficult balance to strike, hopefully my choices have been the right ones. One should consider this monograph as a guide on how to introduce oneself to this subject, rather than as a complete and systematic “exposé” of a more traditional format.

Many people are to be thanked here. Some because they have patiently explained to me (in person, at conferences or in writing) many of the beautiful mathematical notions that are presented in this work. Others because they have patiently listened to sometimes overenthusiastic renditions of these same notions. Happily for me, many are also to be counted in both groups, so that I will not discriminate them in my thanks. My first thanks go to my close mathematical family: Nantel Bergeron, Adriano Garsia, and Christophe Reutenauer; as well as those (most often friends) that have had such a profound mathematical impact on me: Persi Diaconis, Sergey Fomin, Ira Gessel, Mark Haiman, André Joyal, Alain Lascoux, I.G. Macdonald, Gian-Carlo Rota, Richard Stanley, and Xavier Viennot. Let me add to these the relevant members of our research center² “Lacim,” as well as visitors and special friends: Marcelo Aguiar, Jean-Christophe Aval, Srecko Brlek, Frédéric Chapoton, Sara Faridi, Anthony Geramita, Alain Goupil, Jim Haglund, Florent Hivert, Christophe Hohlweg, Gilbert Labelle, Luc Lapointe, Bernard Leclerc, Pierre Leroux (the sadly deceased founder of our research center), Claudia Malvenuto, Jennifer Morse, Frédéric Patras, Bruce Sagan, Franco Salviola, Manfred Schocker, Jean-Yves Thibon, Glenn Tesler, Luc Vinet, Stephanie Van Willigenburg, and Mike Zabrocki. They have all knowingly or unknowingly contributed to this project and, together with all the others of the Lacim large community, they have made all this a very enjoyable daily experience. I also want to thank students and postdoctoral fellows that have been closely tied to the study of the relevant material: Riccardo Biagioli, Anouk Bergeron-Brlek, Philippe Choquette, Sylvie Hamel, Aaron Lauve, François Lamontagne, Peter McNamara, and Mercedes Rosas. They are the ones that may have suffered most from my obsessions, even if they have not yet publicly complained. I am very worried that I am forgetting to thank someone here, and I hope this is not your case.

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²Go to the relevant web page if you are anxious to know what the acronym means.