Elliptic Hall Algebra
and
Rational Catalan Combinatorics

Joint with A. Gar西亚
RATIONAL CATALAN COMBINATORICS
\[ D_{k,m} = \frac{1}{k+m} \binom{k+m}{k} \]

\( k \perp m \) coprime

RATIONAL SLOPE

\((k,m)\)-DYCK PATHS
\( \rho_{k,m} = k^{m-1} \)
$$(\kappa, \kappa, m)$$ Padjik Fraguithus

$$35302 \quad \lambda(\alpha) = 11201$$
DERIVATION OF A NEW FORMULA FOR THE NUMBER OF MINIMAL LATTICE PATHS FROM \((0,0)\) TO \((km, kn)\) HAVING JUST \(t\) CONTACTS WITH THE LINE \(my = nx\) AND HAVING NO POINTS ABOVE THIS LINE; AND A PROOF OF GROSSMAN'S FORMULA FOR THE NUMBER OF PATHS WHICH MAY TOUCH BUT DO NOT RISE ABOVE THIS LINE

By M. T. L. Bizley, F.I.A., F.S.S., F.I.S.

Whitworth(1) deals in Chapter \(v\) of Choice and Chance with the problem of finding the number of minimal lattice paths from \((0,0)\) to \((k,k)\) which do not cross the line \(y = x\). By a lattice path is meant a path joining two points with integral coefficients by a line composed of horizontal and vertical steps of unit length. A minimal lattice path from \((0,0)\) to \((x,y)\), say, is a lattice path where the total number of steps is \((x+y)\); in other words, all the steps are onwards. In what follows minimal lattice paths only will be considered, and the words 'minimal lattice' will be omitted.

Although Whitworth deals only with the case where the boundary line (i.e. the line which the path must not cross) is \(y = x\), the more general case of a boundary \(xy = x\) has been solved provided \(x\) is a positive integer(2), (3). The number of paths from \((0,0)\) to \((x, l)\) which may touch but never rise above \(xy = x\) is 

\[
\frac{1}{lx + 1} \left( \begin{array}{c} lx + l \\ l \end{array} \right).
\]

Grossman(4) announced without proof in 1950 a formula for the number of paths from \((0,0)\) to \((km, kn)\) which may touch but never rise above the line \(my = nx\), where \(k\) is a positive integer and \(m\) and \(n\) are coprime positive integers; thus \((km, kn)\) is any point having positive integral coefficients. Grossman's formula is

\[
\sum P_1^k P_2^n \ldots / k_1! k_2! \ldots,
\]

where

\[
F_s = \frac{1}{j(m + n)} \left( \begin{array}{c} jm + jn \\ jm \end{array} \right),
\]
Bizley's Formula

$$\sum_{d \geq 0} a_d b_d \tilde{\gamma}^d = \exp \left( \sum_{k \geq 1} \frac{1}{a+b} \left( \frac{a^k + b^k}{a^k} \right) \tilde{\gamma}^k \right)$$

\[ a \perp b \text{ coprime} \]
Formulas for the Frobenius

\[ P_{R,M}(x) = \sum_{\alpha \in \text{(R,M)-Dyck Path}} h_{\alpha'}(x) \]

\[ P_{R,M}(x) = \frac{1}{R} h_M[R \cdot x] \quad \text{if } a \perp b \text{ coprime} \]
\[ h^m_n \]

Plethystic Notation

\[(f + g)[\text{\(\bigcirc\)}] = f[\text{\(\bigcirc\)}] + g[\text{\(\bigcirc\)}] \]

\[(f \cdot g)[\text{\(\bigcirc\)}] = f[\text{\(\bigcirc\)}] \cdot g[\text{\(\bigcirc\)}] \]

\[ \mathfrak{h}[x] = \frac{\phi_n(x)}{1 - g^n} \]
Our Generalization of Bizley's Formula

\[ \sum_{d \geq 0} \varphi_{ad, bd}(x) z^d = \exp \left( \sum_{k \geq 1} \frac{1}{a} h_{bk} [a_k x] z^{1/k} \right) \]

\( \text{THM} \)

\( a \perp b \) coprime

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The $q$-Frobenius $P_{k,m}(x; q) := \sum_{\alpha} q^{\text{AREA}(\alpha)} b_{\alpha}(x)$ for $(k,m)$-Dyck paths.
REPRESENTATION THEORY

MOTIVATES $\to \mathcal{P}_{k,m}(x; g, t)$

AT PRESENT, THIS IS ONLY UNDERSTOOD FOR SPECIAL VALUES OF $k$, SUCH AS

$k = mm \pm 1$

(LINKED TO THE STUDY OF HILBERT SCHEMES)
Working with the Elliptic Hall Algebra
Burban - Schiffmann - Vasserot

**Burban - Schiffmann**

*On the Hall Algebra of an Elliptic Curve*

*Duke Math J. 2012*

**Schiffmann - Vasserot**

*The Elliptic Hall Algebra, Cherednik Hecke Algebras, and Macdonald Polynomials*

*Duke Math J. 2013*
E. Gorsky

RIM

coprimE

Refined Knot Invariants
and Hilbert Schemes

arXiv:1304.3328
**Our Main Conjecture**

For $k$ and $m \geq 1$

$$E_{k,m} \cdot 1 = \bigwedge^k (x; q, t)$$

Operators of symmetric functions

$$E_{k,m} : \bigwedge^d \rightarrow \bigwedge^{d+m}$$
For $k$ and $m \geq 1$ the $E_{k,m}$ are polynomials in $D_j$

$$D_j f(x) := f[x + M/M] \Omega [-z^x]$$

$$M := (1-t)(1-g)$$

$$\Omega [-z^x] := 1 - e_1(x) z + e_2(x) z^2 - e_3(x) z^3 + \ldots$$
\[ E_{k,m} \in \mathbb{Q}(t)[D_0, D_1, D_2, \ldots] \]

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**Degree \( k \)**

**If** \( m > k \)

\[ E_{k,m} = E_{k,m-k} \bigg| \begin{array}{c}
D_j \mapsto -D_{j+1}
\end{array} \]

**If** \( k > m \)

\[ E_{k,m} = E_{k-n,m} \bigg| \begin{array}{c}
D_j \mapsto \partial_0 y \cdot D_0
\end{array} \]
\[ [A, B] = AB - BA \]

\[
\partial_A^j B := \begin{cases} 
[A, \partial_A^{j-1} B] & \text{if } j > 0 \\
B & \text{if } j = 0
\end{cases}
\]
\[ E_{k,m} \in \mathbb{Q}(q,t)[D_0, D_1, D_2, \ldots] \]

\[ \text{DEGREE } k \]

\[ \text{IF } k = m \]

\[ E_{m,m} := \sum_{\mu \vdash m} \psi(1, q, t) \prod_{k \in \mu} \frac{-1}{\mu^{k-1}} [D_0, D_0^2, D_2] \]

\[ \psi(1, q, t) \text{ ARE SUCH THAT} \]

\[ e_m = \sum_{\mu \vdash m} \psi(1, q, t) \left( \sum_{k=1}^{\mu} \left( \frac{-1}{q^t} \right)^{-k-1} S_{k, m-k} \right) \]
SUPPORTING EVIDENCE

1. Proof for infinite families of cases

2. Proof for $t = 1$ or $t = \frac{1}{q}$

3. Extensive computer experimental data
Fin