On a Conjecture Concerning Littlewood-Richardson Coefficients

F. Bergeron, Lacim, UQAM

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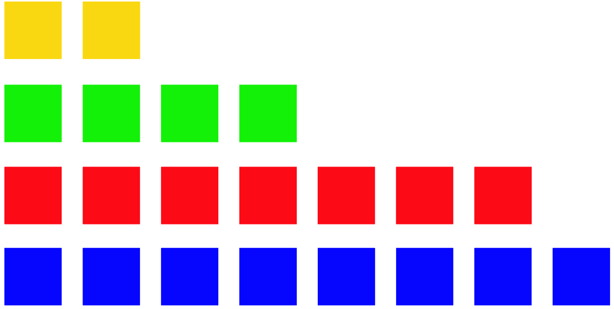
F. Bergeron, Lacim, UQAM
(with R. Biagioli and M. Rosas)

Partitions of integers

$\mu = \mathbf{8742}$



$$\mu \vdash n, \qquad \mu = (\mu_1, \mu_2, \ldots, \mu_k),$$

$$n = |\mu| = \mu_1 + \mu_2 + \ldots + \mu_k, \qquad \mu_1 \geq \mu_2 \geq \cdots \geq \mu_k > 0,$$

We say that k is the *length*, $\ell(\mu)$, of μ.

Symmetric Functions (Polynomials)

Linear combinations of

$$m_\lambda(x_1, x_2, \ldots, x_n) := \sum x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_k}^{\lambda_k},$$

the sum being over all possible choices of distinct i_ℓ's.

Example: for $n = 3$,

$$m_2 + 2\,m_{11} = x_1^2 + x_2^2 + x_3^2 + 2\,(x_1x_2 + x_1x_3 + x_2x_3)$$

Schur Functions

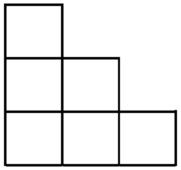
“Recall” that

$$S_\mu = \sum_{T \text{ of shape } \mu} X_T$$

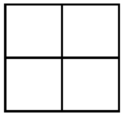
where for

$$T = \begin{array}{|c|c|c|} \hline \mathbf{4} \\ \hline \mathbf{2} & \mathbf{5} \\ \hline \mathbf{1} & \mathbf{2} & \mathbf{2} \\ \hline \end{array} \qquad \mapsto \qquad X_T = x_1 x_2^3 x_4 x_5$$

T semi-standard of shape $\mu = 321 =$



Example of Schur Functions

We want to *compute* S_μ, for $\mu = 22 =$ . For this, choose $a < b < c < d$, positive integers.

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The possible semi-standard tableaux of shape μ are:

$$\begin{array}{|c|c|}\hline c & d \\ \hline a & b \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline b & d \\ \hline a & c \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline b & c \\ \hline a & a \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline b & c \\ \hline a & b \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline c & c \\ \hline a & b \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline b & b \\ \hline a & a \\ \hline\end{array}$$

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\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow
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\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
x_a x_b x_c x_d & x_a x_b x_c x_d & x_a^2 x_b x_c & x_a x_b^2 x_c & x_a x_b x_c^2 & x_a^2 x_b^2
\end{array}$$

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$$\downarrow_{\Sigma} \qquad \downarrow_{\Sigma} \qquad \downarrow_{\Sigma} \qquad \downarrow_{\Sigma} \qquad \downarrow_{\Sigma} \qquad \downarrow_{\Sigma}$$

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\downarrow\Sigma & \downarrow\Sigma & \multicolumn{3}{c}{\underbrace{\downarrow\Sigma \qquad\qquad \downarrow\Sigma \qquad\qquad \downarrow\Sigma}_{m_{211}}} & \downarrow\Sigma \\
m_{1111} & m_{1111} & & & & m_{22}
\end{array}
$$

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m_{1111} & m_{1111} & m_{211} & m_{22}
\end{array}$$

Hence $S_{22} = 2\,m_{1111} + m_{211} + m_{22}$

Schur Functions:
they form a basis

$$\begin{aligned}
S_4 &= m_4 + m_{31} + m_{22} + m_{211} + m_{1111} \\
S_{31} &= m_{31} + m_{22} + 2\,m_{211} + 3\,m_{1111} \\
S_{22} &= m_{22} + m_{211} + 2\,m_{1111} \\
S_{211} &= m_{211} + 3\,m_{1111} \\
S_{1111} &= m_{1111}
\end{aligned}$$

Littlewood-Richardson coefficients

For μ partition of m and ν partition of k,

$$S_\mu S_\nu = \sum_{\theta \vdash m+k} c_{\mu,\nu}^{\theta} S_\theta$$

with $c_{\mu,\nu}^{\theta}$ non negative integers.

Examples of Littlewood-Richardson coefficients

$$\begin{aligned} S_1 S_3 &= S_4 + S_{31} \\ S_2 S_2 &= S_4 + S_{31} + S_{22} \end{aligned}$$

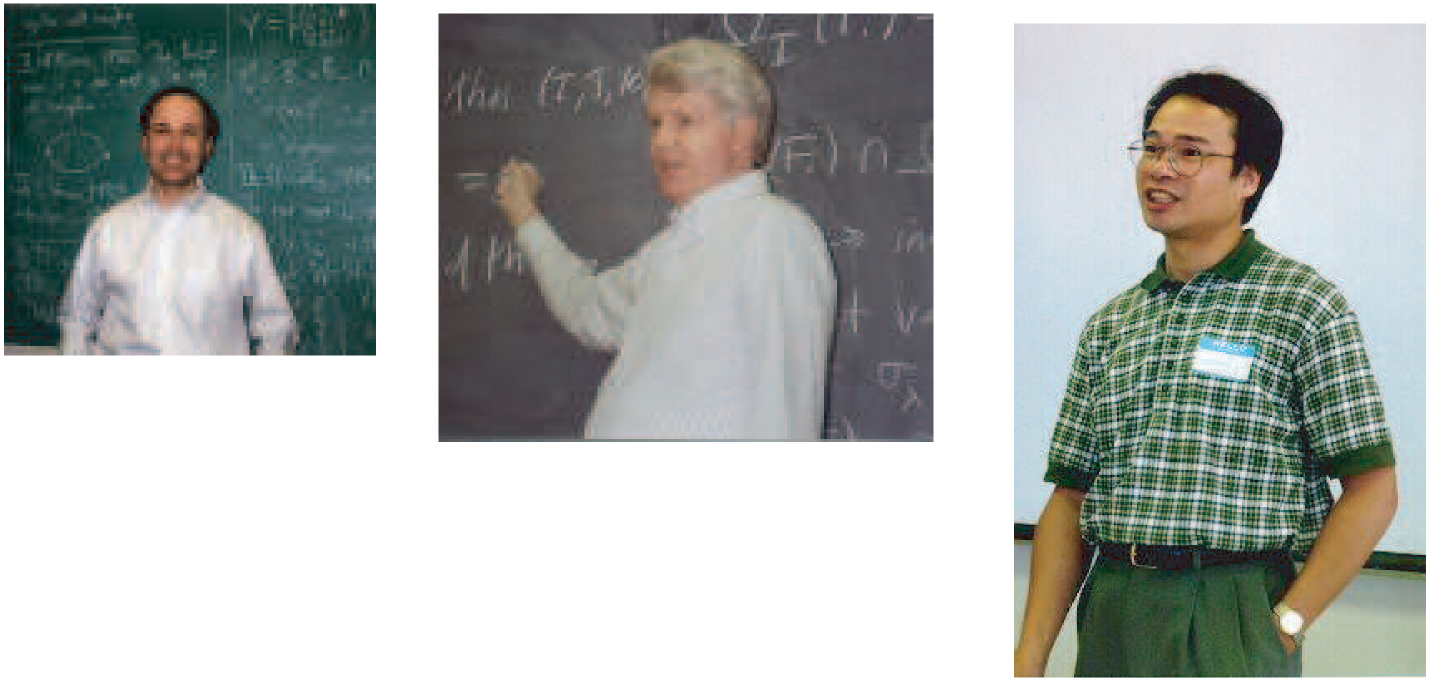
Origin of the problem

S. Fomin, W. Fulton, Chi-Kwong Li, and Yiu-Tung Poon, *Eigenvalues, Singular Values, and Littlewood-Richardson Coefficients*, to appear in Amer. J. Math. Also in arXiv:math.AG/030130.

They give Horn-type inequalities.

A. Horn, *Eigenvalues of sums of Hermitian matrices*, Pacific J. Math., **12** (1962), 225–241.

Fomin, Fulton, Li and Poon.



X

Singular values of matrices

The *singular values* of a complex ℓ by m matrix X are the positive square roots of eigenvalues of the positive semidefinite matrix X^*X, where X^* denotes the conjugate transpose of X.

Typical result of FFLP

For $2\,p \leq n$, let $x_1 \geq \cdots \geq x_p$ be the singular values of a p by $n-p$ matrix X, then the singular values $z_1 \geq \cdots \geq z_n$ of any symmetric matrix

$$Z = \begin{pmatrix} * & X \\ X^* & * \end{pmatrix}$$

are such that

$$2\sum_{k \in K} x_k \leq \sum_{i \in I} z_{2i-1} + \sum_{j \in J} z_{2j}$$

where $(I, J, K) \in LR_r^p$ are triples for which some Littlewood-Richardson coefficient does not vanish.

Transformation of FFLP

For partitions μ and ν, define

$$\begin{aligned}\lambda_k &:= \mu_k - k + \#\{j \mid \nu_j - j \geq \mu_k - k\};\\ \rho_j &:= \nu_j - j + 1 + \#\{k \mid \mu_k - k > \nu_j - j\}.\end{aligned}$$

Pad μ and ν with 0's, if needed, to make them of same length.

Combinatorial properties of FFLP's transformation

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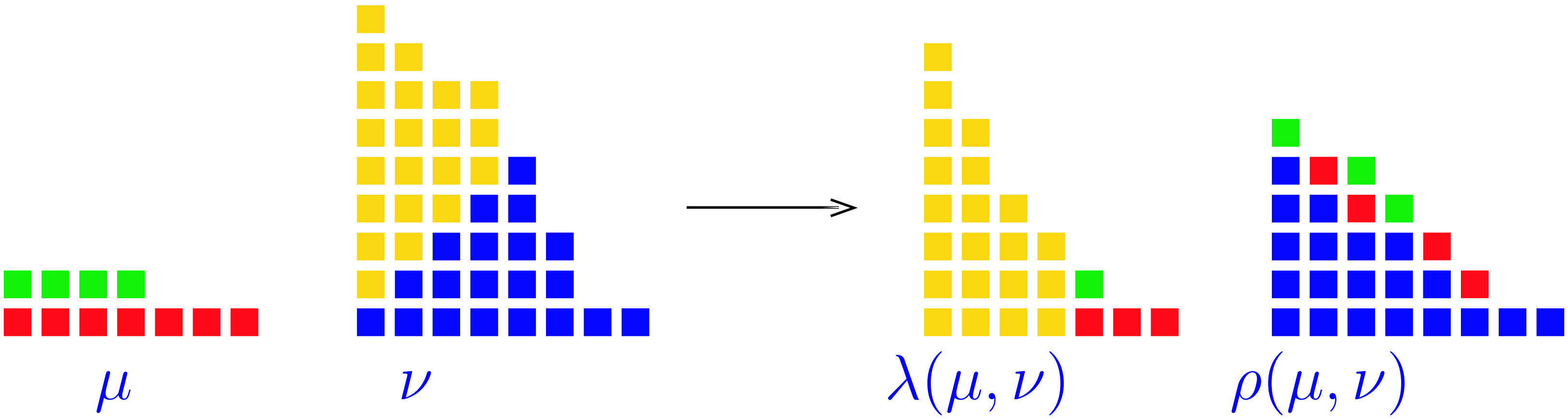
1. (FFLP). $|\lambda| + |\rho| = |\mu| + |\nu|$.

Understanding of FFLP's transformation



Conjecture (Fomin-Fulton-Li-Poon). *For any pair of partitions (μ, ν), the symmetric function*

$$S_{\lambda(\mu,\nu)} S_{\rho(\mu,\nu)} - S_\mu S_\nu$$

is Schur-positive.

Reformulation in term of Littlewood-Richardson coefficients

FFLP's conjecture states that

$$c^{\theta}_{\lambda,\rho} \geq c^{\theta}_{\mu,\nu}$$

writing λ for $\lambda(\mu, \nu)$ and ρ for $\rho(\mu, \nu)$.

Link to algebraic geometry

FFLP's conjecture implies that the intersection of some Schubert cells

$$\Omega^0_{\tau(\lambda,\mu)}(A_\bullet), \qquad \Omega^0_{\tau(\lambda,\mu)}(B_\bullet), \quad \text{and} \quad \Omega^0_{\tau(\nu,\nu)}(C_\bullet)$$

are transversal at each points of a certain form. Here, $A_\bullet$, $B_\bullet$ and $C_\bullet$ are complete flags in a vector space of the form $V \oplus V$, and $\Omega^0_\lambda(E_\bullet)$ corresponds to some subset (variety) of the *Grassmannian* $G(p, V \oplus V)$, namely p-dimensionnal subspaces that intersect $E_\bullet$ in some given (by λ) list of dimensions.

Implication

This algebraic-geometrical fact would, in turn, give a new proof of a result of Carré and Leclerc, on an inequality involving Littlewood-Richardson coefficients, which is a crucial element in the proof of their results.

Easy example

Let $a > b$ be two positive integers, one check that

$$\lambda((a),(b)) = (a-1), \qquad \text{and} \qquad \rho((a),(b)) = (b+1).$$

and the conjecture is true since

$$\begin{aligned} S_\lambda S_\rho - S_\mu S_\nu &= S_{a-1}S_{b+1} - S_a S_b \\ &= \det \begin{pmatrix} S_{a-1} & S_a \\ S_b & S_{b+1} \end{pmatrix} \\ &= S_{(a-1,b+1)} \end{aligned}$$

by Jacobi-Trudi's formula for Schur functions.

Explicit example

Recall that

$$\begin{aligned} S_1 S_3 &= S_4 + S_{31} \\ S_2 S_2 &= S_4 + S_{31} + S_{22} \end{aligned}$$

Hence, for $a = 3$ and $b = 1$, we have

$$\begin{aligned} S_\lambda S_\rho - S_\mu S_\nu &= \det \begin{pmatrix} S_2 & S_3 \\ S_1 & S_2 \end{pmatrix} \\ &= S_{22} \end{aligned}$$

Other Combinatorial properties of FFLP's transformation

3. (BBR). $\lambda'(\nu', \mu') = \lambda(\mu, \nu)$ and $\rho'(\nu', \mu') = \rho(\mu, \nu)$, where as usual μ' stands for the conjugate of μ.

Other Combinatorial properties of FFLP's transformation

3. (BBR). $\lambda'(\nu', \mu') = \lambda(\mu, \nu)$ and $\rho'(\nu', \mu') = \rho(\mu, \nu)$, where as usual μ' stands for the conjugate of μ.

4. (BBR). If $\alpha \subseteq \mu$ and $\beta \subseteq \nu$, then

$$\lambda(\alpha, \beta) \subseteq \lambda(\mu, \nu) \qquad \text{and} \qquad \rho(\alpha, \beta) \subseteq \rho(\mu, \nu).$$

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5. (BBR). There is a simple recursive description of the transformation, adding one cell at a time.

Fixed points

An observation of FFLP: The pair (μ, ν) is fixed by the transformation, iff

$$\mu_1 \geq \nu_1 \geq \mu_2 \geq \nu_2 \geq \cdots \geq \mu_k \geq \nu_k \geq \cdots$$

In particular, (μ, μ) is fixed.

Using fixed points

If (α, β) and (γ, δ) are two fixed points with

$$\alpha \subseteq \gamma \qquad \text{and} \qquad \beta \subseteq \delta,$$

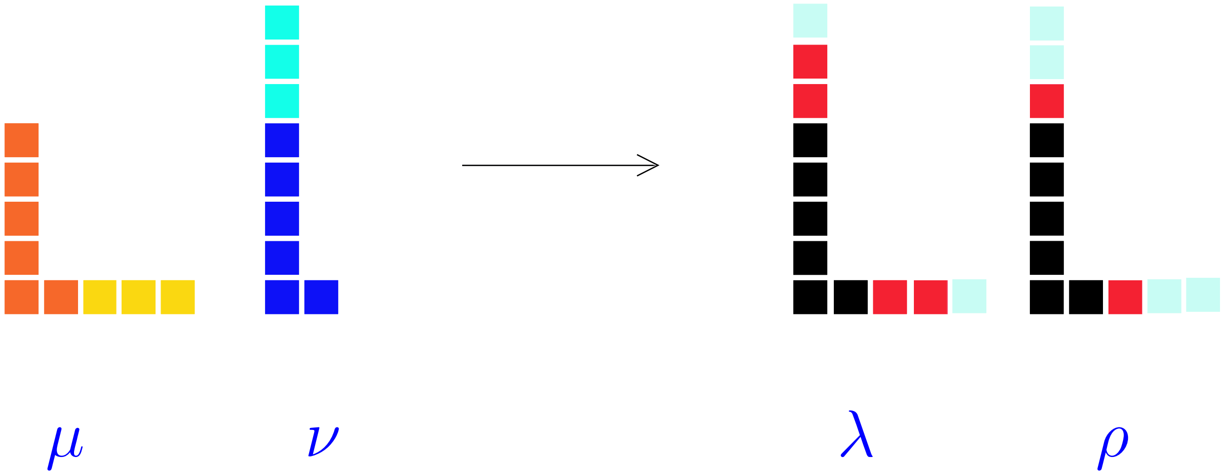
then whenever

$$\alpha \subseteq \mu \subseteq \gamma \qquad \text{and} \qquad \beta \subseteq \nu \subseteq \delta,$$

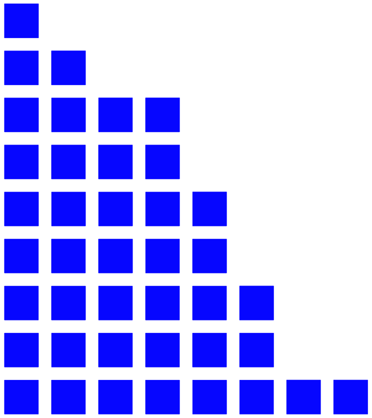
we also have

$$\alpha \subseteq \lambda(\mu, \nu) \subseteq \gamma \qquad \text{and} \qquad \beta \subseteq \rho(\mu, \nu) \subseteq \delta.$$

Pair of hooks

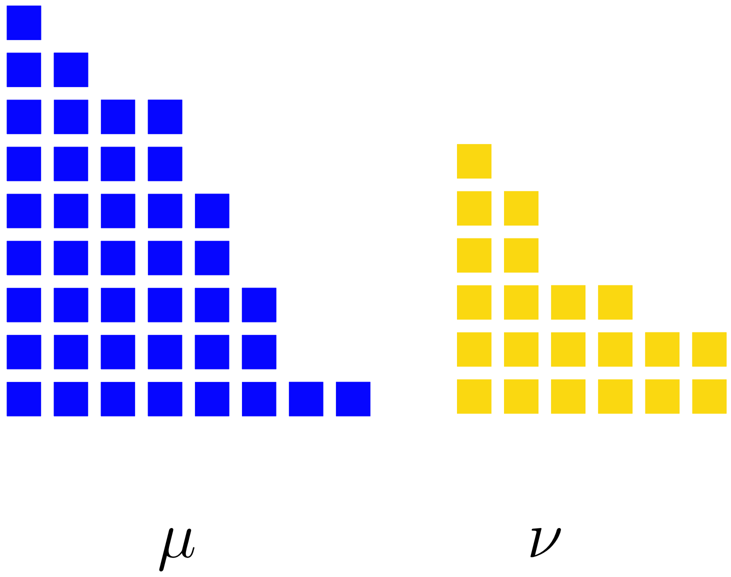


Skew partitions



μ

Skew partitions



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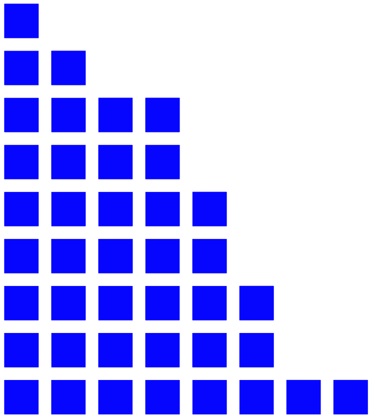
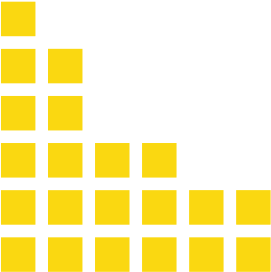
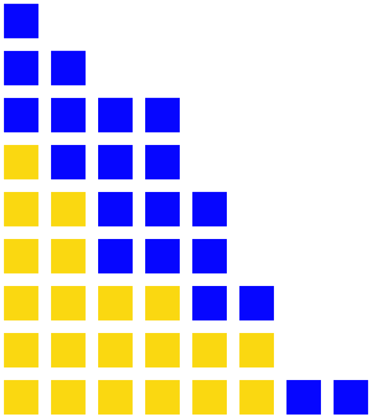


μ

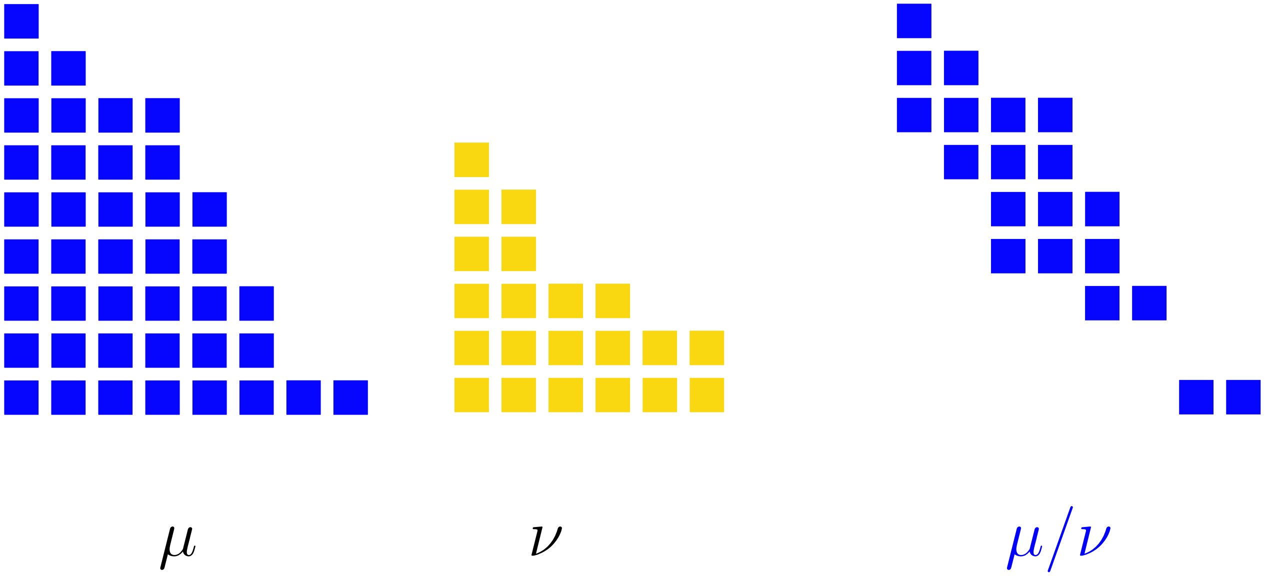


ν



$\nu \subseteq \mu$

Skew partitions



Skew Schur

Same as before

$$S_{\mu/\alpha} = \sum_{T \text{ of shape } \mu/\alpha} X_T$$

where for

$$T = \begin{array}{|c|cccc} \hline \mathbf{4} & \multicolumn{1}{|c}{} & & & \\ \hline & \multicolumn{1}{|c|}{\mathbf{2}} & \multicolumn{1}{c|}{\mathbf{5}} & & \\ \cline{2-5} & & \multicolumn{1}{|c|}{\mathbf{1}} & \multicolumn{1}{c|}{\mathbf{2}} & \multicolumn{1}{c|}{\mathbf{5}} \\ \cline{3-5} \end{array} \qquad \mapsto \qquad X_T = x_1 x_2^2 x_4 x_5^2$$

Property 3. allows
an extension of the conjecture

Conjecture (Bergeron-Biagioli-Rosas). *For any skew shapes* μ/α *and* ν/β*, let*

$$\lambda := \lambda(\mu,\nu)/\lambda(\alpha,\beta) \qquad \text{and} \qquad \rho := \rho(\mu,\nu)/\rho(\alpha,\beta),$$

then the symmetric function

$$S_\lambda S_\rho - S_{\mu/\alpha} S_{\nu/\beta}$$

is Schur-positive.

A sample of new results

1. FFLP's conjecture holds for the pair (μ, ν) if and only if it holds for the pair (ν', μ').

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2. For any p there exists an explicit m such that, FFLP's conjecture holds for any (μ, ν) with $\ell(\mu) \leq p$ and $\nu \subseteq (p^p)$ iff it holds for those (μ, ν) with $\mu \subseteq (m^p)$ and $\nu \subseteq (p^p)$.

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3. For any (μ, ν), $h_\lambda h_\rho - h_\mu h_\nu$ is Schur-positive.
4. For (μ, ν) any pair of hook shapes, FFLP's conjecture holds. Moreover, $c^\theta_{\mu,\nu} \leq c^\theta_{\lambda,\rho} \leq 2$.

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3. For any (μ, ν), $h_\lambda h_\rho - h_\mu h_\nu$ is Schur-positive.
4. For (μ, ν) any pair of hook shapes, FFLP's conjecture holds. Moreover, $c^\theta_{\mu,\nu} \leq c^\theta_{\lambda,\rho} \leq 2$.
5. The conjecture holds for many other infinite families of pairs (some involving the skew case).

Recursive description of the transformation

Some notation:

$\alpha \rightarrow_{\ell} \mu$: μ is obtained from α by adding one cell in line ℓ;

$\alpha \rightarrow^{k} \mu$: μ is obtained from α by adding one cell in column k.

For example:

Recursive description of the transformation

For a given ℓ, let $\alpha \to_\ell \mu$,

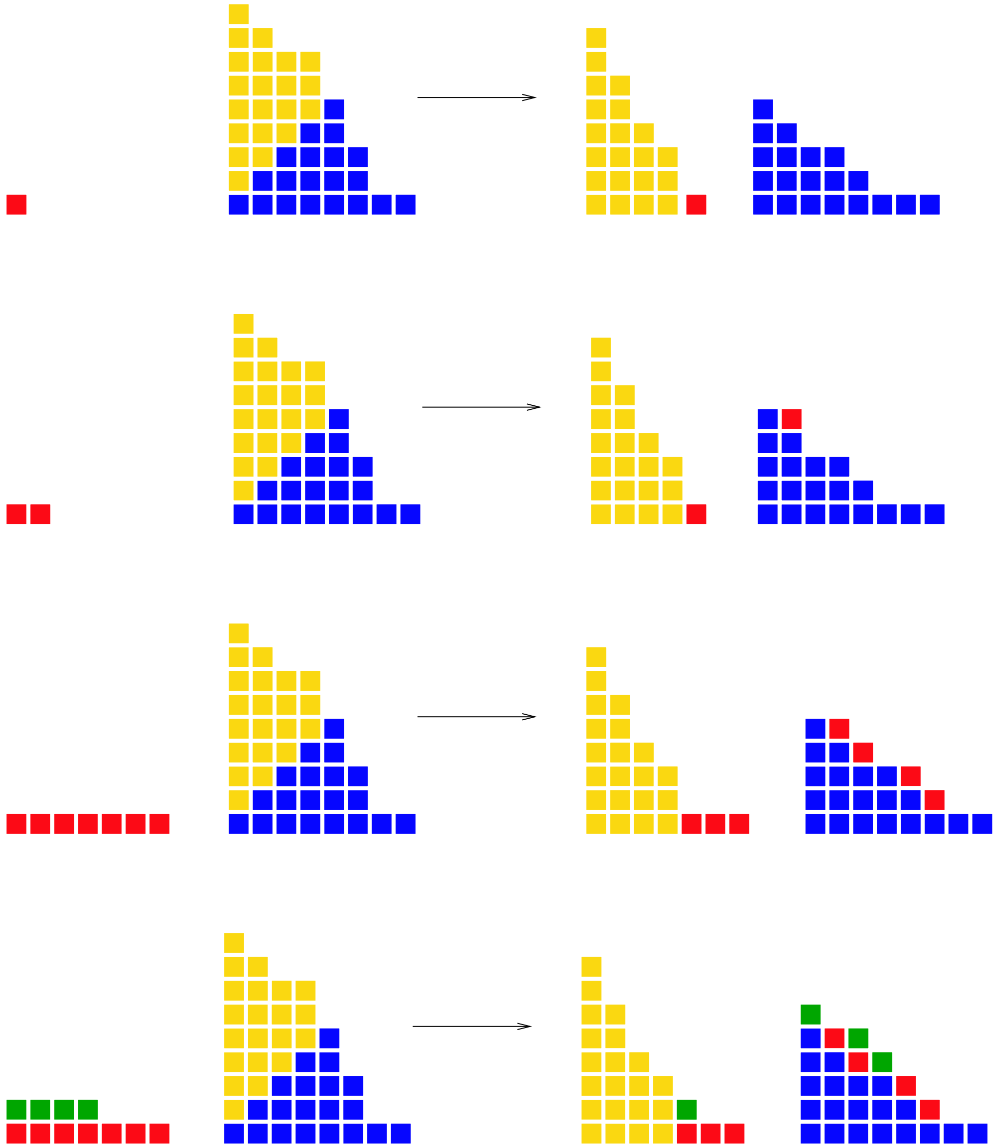
a) if $\nu_j - j = \alpha_\ell - \ell$ for some j, then

$$\lambda(\alpha,\nu) = \lambda(\mu,\nu) \qquad \text{and} \qquad \rho(\alpha,\nu) \to^{\mu_\ell} \rho(\mu,\nu),$$

b) otherwise

$$\lambda(\alpha,\nu) \to_\ell \lambda(\mu,\nu) \qquad \text{and} \qquad \rho(\alpha,\nu) = \rho(\mu,\nu).$$

Recursive description of the transformation



Transformation of tableaux

$$\left(\begin{array}{|c|c|c|c|c|} \hline 26 & \multicolumn{4}{c}{} \\ \cline{1-3} 22 & 23 & 24 & \multicolumn{2}{c}{} \\ \hline 16 & 17 & 18 & 19 & 20 \\ \hline 9 & 10 & 11 & 12 & 13 \\ \hline \end{array} \quad , \quad \begin{array}{|c|c|c|c|c|c|c|c|} \cline{1-1} 25 & \multicolumn{7}{c}{} \\ \cline{1-1} 21 & \multicolumn{7}{c}{} \\ \cline{1-2} 14 & 15 & \multicolumn{6}{c}{} \\ \hline 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline \end{array} \right)$$

transforms into

$$\left(\begin{array}{|c|c|c|c|c|} \cline{1-1} 26 & \multicolumn{4}{c}{} \\ \cline{1-2} 22 & 24 & \multicolumn{3}{c}{} \\ \cline{1-4} 16 & 17 & 19 & 20 & \multicolumn{1}{c}{} \\ \hline 9 & 10 & 11 & 12 & 13 \\ \hline \end{array} \quad , \quad \begin{array}{|c|c|c|c|c|c|c|c|} \cline{1-1} 25 & \multicolumn{7}{c}{} \\ \cline{1-2} 21 & 23 & \multicolumn{6}{c}{} \\ \cline{1-3} 14 & 15 & 18 & \multicolumn{5}{c}{} \\ \hline 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline \end{array} \right)$$

Transformation of semi-standard tableaux

Transform an increasing sequence of partition pairs

$$(0,0) \subset \cdots \subset (\mu^{(i)}, \nu^{(i)}) \subset \cdots \subset (\mu^{(k)}, \nu^{(k)})$$

into the sequence

$$(0,0) \subset \cdots \subset (\lambda^{(i)}, \rho^{(i)}) \subset \cdots \subset (\lambda^{(k)}, \rho^{(k)})$$

where $\lambda^{(i)} = \lambda(\mu^{(i)}, \nu^{(i)})$ and $\rho^{(i)} = \rho(\mu^{(i)}, \nu^{(i)})$.

This is not injective.

Monomial positivity conjecture

Conjecture. *For any pair of (skew) partitions* (μ, ν), *the symmetric function*

$$S_{\lambda(\mu,\nu)}S_{\rho(\mu,\nu)} - S_\mu S_\nu$$

is m-positive.

This is implied by the previous conjecture.

An injective version of the transformation for semi-standard tableaux would prove this conjecture.

This is the end for now