

Is $\mathbb{Q}\text{Sym}_n^{\kappa}$ free over Sym_n ?

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QSym ^{κ} _{n}

As we have seen last Monday, Florent Hivert has introduced a κ -*threshold* local action of \mathbb{S}_n on polynomials. For κ in \mathbb{N} :

$$(i, i+1)_{\kappa}^* x_i^a x_{i+1}^b := \begin{cases} x_i^a x_{i+1}^b & \text{if both } a \geq \kappa \text{ and } b \geq \kappa, \\ x_{i+1}^a x_i^b & \text{otherwise.} \end{cases}$$

Recall that he defines κ -*quasi-symmetric polynomials* to be those that are fixed under this action:

$$\sigma_{\kappa}^* p(\mathbf{x}) = p(\mathbf{x})$$

in $\mathbb{Q}[\mathbf{x}] = \mathbb{Q}[x_1, x_2, \dots, x_n]$. It is natural to identify QSym_n^0 with $\mathbb{Q}[\mathbf{x}]$, and here the action is trivial.

Hiver 2001–02 Conjecture/Theorem

- 1) In **hiver 2001–02**, F. B. and C. Reutenauer conjectured that \mathbf{QSym}_n is a free \mathbf{Sym}_n -module (here \mathbf{Sym}_n is the ring of symmetric polynomials). They showed that this implies that the *Hilbert series*:

$$\varphi_n(q) := \sum_{d \geq 0} \dim(\pi_d R_n) q^d$$

of $R_n := \mathbf{QSym}_n / \mathbf{Sym}_n$, is such that

$$\varphi_n(q) = \varphi_{n-1}(q) + q^n ([n]!_q - \varphi_{n-1}(q))$$

and they conjectured an explicit basis for the above quotient.

- 2) The **hiver 2001–02** conjecture was shown to be true in late 2002 by A. Garsia and N. Wallach. But the basis of F.B. and C.R. still remains conjectural.

Hivert's 2004–05 Conjecture

Hivert conjectured last Monday (in fact clearly before then) that QSym_n^κ is free as a Sym_n -module, and showed that this implies that the Hilbert series of the quotient $R_n^\kappa := \text{QSym}_n^\kappa / I_n$ (with $I_n = \text{Sym}_n^+ \text{QSym}_n^\kappa$) is given by the formula:

$$\varphi_n^\kappa(q) = \sum_{\sigma \in \mathcal{S}_n} q^{\text{maj}(\sigma) + \kappa(n - \text{fix}(\sigma))}$$

Garsia has observed that we have the **wonderful recursion**

$$\varphi_n^\kappa(q) = \varphi_{n-1}^\kappa(q) + q^n(\varphi_n^{\kappa-1}(q) - \varphi_{n-1}^\kappa(q))$$

However his joint efforts with Wallach have apparently not yet allowed the extension of all aspects of their $\kappa = 1$ approach to the new context.

New recurrence

Trying to find a basis for the quotient R_n^κ , we were led to the recurrence

$$\begin{aligned}\varphi_n^\kappa(q) &= \varphi_{n-1}^\kappa(q) + \psi_n^\kappa(q) \\ \psi_n^\kappa(q) &= [n-1]_q (q^{2\kappa+n-1} \varphi_{n-2}^\kappa(q) + q^\kappa \psi_{n-1}^\kappa(q))\end{aligned}$$

with $\varphi_0^\kappa(q) = \varphi_1^\kappa(q) = 1$, and $\psi_0^\kappa(q) = \psi_1^\kappa(q) = 0$. As we will see, this suggests a new avenue for the construction of a basis for all R_n^κ .

For example, we easily see that $\psi_2^\kappa(q) = q^{2\kappa+1}$ and hence $\varphi_2^\kappa(q) = 1 + q^{2\kappa+1}$, so that

$$\psi_3^\kappa(q) = (1+q)(q^{2\kappa+2} + q^\kappa q^{2\kappa+1})$$

and thus

$$\varphi_3^\kappa(q) = 1 + q^{2\kappa+1} + q^{2\kappa+2} + q^{2\kappa+3} + q^{3\kappa+1} + q^{3\kappa+2}$$

Basis of QSym_n^κ

Recall from Hivert's talk that a basis of QSym_n^κ is given by the set of *monomial* κ -quasi-symmetric polynomials

$$M_{\omega, \lambda}(\mathbf{x}) = \sum_{\mathbf{a} \subseteq \mathbf{x}} \mathbf{a}^\omega m_\lambda(\mathbf{x} - \mathbf{a})$$

with \mathbf{a} varying among the $\ell(\omega)$ -subsets of \mathbf{x} , with induced order.

Here, ω is a *composition* with parts $\geq \kappa$, and λ is a *partition* with parts $< \kappa$, and $\ell(\omega) + \ell(\lambda) \leq n$.

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Elements of QSym_n^κ

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with \mathbf{a} varying among the $\ell(\omega)$ -subsets of \mathbf{x} , with induced order. Here, ω is a *composition* with parts $\geq \kappa$, and λ is a *partition*.

For example

With $\mathbf{x} = x_1 + x_2 + \dots + x_5$,

$$M_{be,21}(\mathbf{x}) = x_1^b x_2^e m_{21}(x_3, x_4, x_5) + x_1^b x_3^e m_{21}(x_2, x_4, x_5) \\ \dots + x_2^b x_3^e m_{21}(x_1, x_4, x_5) + \dots$$

Our **convention** here (and in the sequel) is to write

$$a = \kappa, b = \kappa + 1, c = \kappa + 2, \dots$$

Thus, composition with parts $\geq \kappa$ become words.

κ -independent (almost)-basis of R_n^κ

We “follow” the recurrence to construct a family of composition. In the non-commutative algebra $\mathbb{N}(a, b, c, \dots)^*$, set

$$A_n := A_{n-1} + B_n$$

$$B_n := z(a + b + c + \dots)A_{n-2} + (a + b + c + \dots)B_{n-1}$$

with $A_0 = A_1 = 1$ (the empty word), and $B_0 = B_1 = 0$, we get

- $A_2 = 1 + ba,$
- $A_3 = A_2 + (ca + cb + aba + bba),$
- $A_4 = A_3 + (da + db + dc + daba + dbba + dcba) + (aca + bca + cca + acb + bcb + ccb + aaba + baba + caba + abba + bbba + cbba),$

Clearly using the map $a \mapsto q^\kappa$, $b \mapsto q^{\kappa+1}$, $c \mapsto q^{\kappa+2}$, \dots ; we obtain $\varphi_n^\kappa(q)$ as the degree weighted enumerator of the set

$$\mathcal{A}_n := \{M_\omega \mid \omega \in A_n\}.$$

Explicit description of A_n

The compositions

$$\omega = \alpha_1 \alpha_2 \dots \alpha_\ell$$

that appear in the set A_n are exactly those that satisfy the following conditions

- 1) $\ell \leq n$,
- 2) $\kappa \leq \alpha_i \leq \kappa + n - i$, for $1 \leq i \leq n$,
- 3) The maximal strictly decreasing suffix of ω is of even length.

Observe that, for $\kappa = 0$, we get the **classical Artin basis !!!!** of the space of coinvariants of S_n : $\mathbb{Q}[\mathbf{x}]/I_n$. More precisely, let

$$\Theta(M_\omega) := \mathbf{x}^{\theta(\omega)}, \quad \text{with} \quad \theta(\omega) = \omega - \kappa^{\ell(\omega)}.$$

Then the classical Artin basis is precisely $\{\Theta(M_\omega) \mid \omega \in A_n\}$.

The problem, when $n = 4$

$$M_{bca} - M_{acb} = s_{2211}M_0 - s_{1111}M_{ba} + s_2M_{ca} - s_1M_{cb} - s_1M_{da} + M_{db}$$

We need only replace M_{acb} by M_{cab} to get a basis that works.

Partition-free reduction

We first show that R_n^κ is generated by the set

$$\{M_\omega \mid \omega \in \{a, b, c, \dots\}^*\},$$

with λ reduced to the empty partition.

First, one easily check that

$$M_\omega m_\lambda = cM_{\omega, \lambda} + \sum_{\ell(\mu) < \ell(\lambda)} c_{\omega', \mu} M_{\omega', \mu}$$

It immediately follows that

$$M_{\omega, \lambda} \equiv \sum_{\ell(\omega') = \ell(\omega)} d_{\omega'} M_{\omega'} \pmod{I_n}$$

Graded dimension

Recall that

$$\varphi_n(q) = \sum_{\sigma \in \mathbb{S}_n} q^{\text{maj}(\sigma) + \kappa(n - \text{fix}(\sigma))}$$

Graded dimension, accounting for length

Set

$$\varphi_n(q, r) = \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma)} r^{n - \text{fix}(\sigma)}$$

Graded dimension, accounting for length

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$$\varphi_n(q, r) = \sum_{\sigma \in \mathbb{S}_n} q^{\text{maj}(\sigma)} r^{n - \text{fix}(\sigma)}$$

Then we have

$$\varphi_n(q, r) = \varphi_{n-1}(q, r) + \psi_n(q, r)$$

$$\psi_n(q, r) = [n-1]_q (r^2 q^{n-1} \varphi_{n-2}(q, r) + r \psi_{n-1}(q, r))$$

Graded dimension, accounting for length

Set

$$\varphi_n(q, r) = \sum_{\sigma \in \mathbb{S}_n} q^{\text{maj}(\sigma)} r^{n - \text{fix}(\sigma)}$$

Then we have

$$\varphi_n(q, r) = \varphi_{n-1}(q, r) + \psi_n(q, r)$$

$$\psi_n(q, r) = r[n-1]_q (r q^{n-1} \varphi_{n-2}(q, r) + \psi_{n-1}(q, r))$$

$$A_n^r := A_{n-1}^r + B_n^r$$

$$B_n^r := (a + b + c + \dots)(z A_{n-2}^r + B_{n-1}^r)$$

The case $n = 3$, with $\kappa \geq 1$

0) For any $t \geq \kappa$, M_t and M_{tt} are both clearly in Sym_n .

1) $M_{st} + M_{ts} \in \text{Sym}_n$,

2) when $s > t = \kappa + 1$, we have

$$M_{st} - (s_{ij}M_{cb} - s_{i11}M_{ba}) \in \text{Sym}_n,$$

3) when $s > t = \kappa + 2$, we have

$$M_{st} - (s_{ij}M_{cb} - s_{ij1}M_{ca}) \in \text{Sym}_n,$$

4) and when $s > t > \kappa + 2$, we have

$$M_{st} - (s_{ij}M_{cb} - s_{ij1}M_{ca} + s_{ij2}M_{ba}) \in \text{Sym}_n.$$

Where we have written i for $s - c$, and j for $t - b$.

Continuing with $n = 3$

We have thus checked that all M_ω , with $\ell(\omega) \leq 2$, reduce to the basis \mathcal{A}_3 modulo the ideal.

When $\ell(\omega) = 3$, we can suppose that at least one of the parts of ω is equal to κ , since otherwise we have $M_\omega = m_{111}M_{\omega-111}$ which is clearly in the ideal.

Moreover we need only consider reductions for $M_{st\kappa}$ and $M_{\kappa st}$, since

$$M_{st\kappa} + M_{s\kappa t} + M_{\kappa st} \equiv_2 M_{st}M_\kappa$$

modulo Sym_n -linear combinations of M_ω 's with $\ell(\omega) < 3$. Let us illustrate, by giving one of these reductions

$$M_{st\kappa} \equiv_2 (s_{i+1,j} - s_{i,j,1})M_{bba} + \frac{1}{2}(s_{i+1,j+1} - s_{i,j+1,1})M_{aba}$$

with i and j , $i > j$.

General case

What is needed to generalize all this is a basis \mathcal{B}_n that is independent of κ . Our construction of \mathcal{A}_n above has to be slightly modified. For instance, when $n = 4$, we need only replace bca (or acb) by cab , and then the result is a basis with the needed property.

We then proceed as for $n = 3$, successively reducing M_ω with at most m parts to M_ν with at most $m - 1$ parts. These reductions are the same for all $\kappa \geq 1$, and we already know that they are possible (and unique) for $\kappa = 1$.

We get a unique decomposition of partition-free elements of QSym_n^κ in such that

$$M_\omega = \sum_{M_\nu \in \mathcal{B}_n \setminus \{1\}} f_\nu M_\nu \in \text{Sym}_n,$$

with the f_ν 's independent of κ .

Diagonally Quasi-symmetric

A κ -bicomposition of a bi-integer $\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ is a sequence of bi-integers:

$$\omega = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix}, \quad \text{with } a_i + b_i \geq \kappa,$$

such that

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} + \cdots + \begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

A κ -bipartition of $\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$, is a bicomposition of $\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ such that

$$\begin{pmatrix} a_i \\ b_i \end{pmatrix} \geq \begin{pmatrix} a_{i+1} \\ b_{i+1} \end{pmatrix} \quad \text{and} \quad 0 < a_i + b_i < \kappa.$$

Here, we use **lexicographic** ordering to compare bi-integers.

We denote $(\mathbf{x}, \mathbf{y})^\omega$, the monomial

$$(\mathbf{x}, \mathbf{y})^\omega := x_1^{a_1} y_1^{b_1} x_2^{a_2} y_2^{b_2} \cdots x_n^{a_n} y_n^{b_n}$$

DQSym_n^κ the ring of diagonally κ -quasi-symmetric polynomials

The ring DQSym_n^κ is the linear span of the polynomials

$$M_{\omega,\lambda}(\mathbf{x}, \mathbf{y}) = \sum_{(\mathbf{a}, \mathbf{b}) \subseteq (\mathbf{x}, \mathbf{y})} (\mathbf{a}, \mathbf{b})^\omega m_\lambda(\mathbf{x} - \mathbf{a}, \mathbf{y} - \mathbf{b})$$

with the (\mathbf{a}, \mathbf{b}) 's running over $l(\omega)$ -subsets of the pairs (x_i, y_i) , with induced order. Here, ω and λ are respectively κ -bicompositions and κ -bipartitions, with $l(\omega) + l(\lambda) \leq n$.

As “usual”, the m_λ 's denote the *monomial diagonally symmetric* polynomials.

The quotient of DQSym_n^κ , by the ideal generated by diagonally symmetric polynomials, can be studied in a similar manner. This is ongoing work.